Exercises

1.5.1. For a locally integrable function f on $\mathbf{R}^n$, $b \in \mathbf{C} \setminus \{0\}$, $\lambda > 0$, and $x_0 \in \mathbf{R}^n$ define the operations $f_\lambda(x) = \lambda^{-n} f(\lambda^{-1}x)$ (L^1 dilation), $\tau^{x_0} f(x) = f(x - x_0)$ (translation), and $\widetilde{f}(x) = f(-x)$ (reflection) for all $x \in \mathbf{R}^n$. Prove the following:

1. $\mathscr{L}_{bf} = \mathscr{L}_f$.
2. $\mathscr{L}_{\widetilde{f}} = -\mathscr{L}_f = \{-y : \, y \in \mathscr{L}_f\}$.
3. $\mathscr{L}_{\overline{f}} = \mathscr{L}_f$, $\overline{f}$ here denotes complex conjugation.
4. $\mathscr{L}_{\tau^{x_0} f} = x_0 + \mathscr{L}_f = \{x_0 + y : \, y \in \mathscr{L}_f\}$.
5. $\mathscr{L}_{f_\lambda} = \lambda \mathscr{L}_f = \{\lambda y : \, y \in \mathscr{L}_f\}$.
6. $\mathscr{L}_{f \circ A} = A^{-1} \mathscr{L}_f = \{A^{-1} y : \, y \in \mathscr{L}_f\}$, where A is an orthogonal matrix.

Moreover, if g is another locally integrable function, prove that $\mathscr{L}_f \cap \mathscr{L}_g \subseteqq \mathscr{L}_{f+g}$.

1.5.2. Show that for every $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ there is a set E_f of measure zero such that

$$\lim_{\varepsilon \to 0} \frac{1}{|B(x,\varepsilon)|} \int_{B(x,\varepsilon)} \left| f(y) - \frac{1}{|B(x,\varepsilon)|} \int_{B(x,\varepsilon)} f(z)dz \right| dy = 0$$

for all $x \in \mathbf{R}^n \setminus E_f$.

1.5.3. Let f be in $L^p(\mathbf{R}^n)$ for some p satisfying $1 \le p < \infty$. Show that

$$\lim_{\delta \to 0} \frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} |f(y) - f(x)|^p \, dy = 0 \quad \text{for almost all } x \in \mathbf{R}^n.$$

1.5.4. Let g be in $L^p(\mathbf{R}^n)$ for some p satisfying $0 < p < 1$. Show that

$$\lim_{\delta \to 0} \frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} |g(y) - g(x)|^p \, dy = 0 \quad \text{for almost all } x \in \mathbf{R}^n.$$

[*Hint:* For every rational number a there is a set E_a of Lebesgue measure zero such that for $x \in \mathbf{R}^n \setminus E_a$ we have

$$\lim_{\delta \to 0} \frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} |g(y) - a|^p \, dy = |g(x) - a|^p,$$

since the function $y \mapsto |g(y) - a|^p$ is in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$. By considering an enumeration of the rationals, find a set of measure zero E such for $x \notin E$ the preceding limit exists for all rationals a and by continuity for all real numbers a, in particular for $a = g(x)$.]

1.5.5. Given $N \in \mathbf{Z}^+$ and $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$, define the function

$$F_N(f) = \sum_{Q \in \mathscr{D}(N)} \left(\frac{1}{|Q|} \int_Q f(y) \, dy \right) \chi_Q ,$$