$$
\begin{aligned}
\big\|f\big\|_{L^{p,\infty}(\mathbf{R}^n)} &\le \big\|M_d(f)\big\|_{L^{p,\infty}(\mathbf{R}^n)} \\
&\le 2^{n+p+2+\frac{1}{p}}\big\|M_d^{\#}(f)\big\|_{L^{p,\infty}(\mathbf{R}^n)} \\
&\le 2^{n+p+2+\frac{1}{p}}\big\|M_c^{\#}(f)\big\|_{L^{p,\infty}(\mathbf{R}^n)} \\
&\le 2^{n+p+3+\frac{1}{p}}\big\|M_c(f)\big\|_{L^{p,\infty}(\mathbf{R}^n)}.
\end{aligned} \tag{6.5.2}
$$

If in addition $p > 1$, then all of the above inequalities become equivalences.

Proof. Since for every point in $\mathbf{R}^n$ there is a sequence of dyadic cubes shrinking to it, the Lebesgue differentiation theorem yields that for almost every point x in $\mathbf{R}^n$ the averages of the locally integrable function f over the dyadic cubes containing x converge to $f(x)$. Consequently, $|f| \le M_d(f)$ a.e., hence the first inequalities in (6.5.1) and (6.5.2) hold. Theorem 6.4.6 provides the second inequalities in both (6.5.1) and (6.5.2). The third inequalities are trivial, while the last inequalities are consequences of Proposition 6.4.4 (2).

Finally, if $p > 1$, then (1.4.7) and Exercise 1.4.8 provide the missing estimates that reverse all inequalities in (6.5.1) and (6.5.2), respectively. $\square$

For the purposes of the next result we set L^∞_{fin} to be the space of all L^∞ functions that are supported in a set of finite measure.

Theorem 6.5.2. *Let $1 < p_0 < \infty$. Let T be a linear operator that maps $L^{p_0}(\mathbf{R}^n)$ to $L^{p_0,\infty}(\mathbf{R}^n)$ with bound A_0 and $L^\infty_{\text{fin}}(\mathbf{R}^n)$ to $BMO(\mathbf{R}^n)$ with bound A_1. Then for all p with $p_0 < p < \infty$ there is an extension of T on $L^p(\mathbf{R}^n)$ and a constant C_{n,p,p_0} such that for all $f \in L^p(\mathbf{R}^n)$ we have*

$$
\big\|T(f)\big\|_{L^p(\mathbf{R}^n)} \le C_{n,p,p_0} A_0^{\frac{p_0}{p}} A_1^{1-\frac{p_0}{p}} \big\|f\big\|_{L^p(\mathbf{R}^n)}. \tag{6.5.3}
$$

Proof. ~~We consider two cases according to the value of p_0.~~
~~**Case I:** $p_0 > 1$.~~ Define the operator $S(f) = M_c^{\#}(T(f))$ for $f \in L^{p_0} + L^\infty_{\text{fin}}$. It is easy to see that S is a subadditive operator. We first prove that S maps L^∞_{fin} to L^∞. Indeed, for $f \in L^\infty_{\text{fin}}$ we have

$$
\big\|S(f)\big\|_{L^\infty} = \big\|M_c^{\#}(T(f))\big\|_{L^\infty} = \big\|T(f)\big\|_{BMO} \le A_1\big\|f\big\|_{L^\infty}.
$$

Next we show that S maps L^{p_0} to $L^{p_0,\infty}$. For $f \in L^{p_0}$ we have

$$
\begin{aligned}
\big\|S(f)\big\|_{L^{p_0,\infty}} &= \big\|M_c^{\#}(T(f))\big\|_{L^{p_0,\infty}} \\
&\le 2\big\|M_c(T(f))\big\|_{L^{p_0,\infty}} \\
&\le 2\,\frac{3^n p_0}{p_0-1}\big\|T(f)\big\|_{L^{p_0,\infty}} \qquad \text{(by Exercise 1.4.8)} \\
&\le 2\,\frac{3^n p_0}{p_0-1} A_0\big\|f\big\|_{L^{p_0}}.
\end{aligned}
$$

Interpolating between these estimates using Theorem 1.3.4, with $\mathscr{F}$ being the space of finitely simple functions, we deduce

$$\big\|M_c^\#(T(f))\big\|_{L^p} = \big\|S(f)\big\|_{L^p} \le 2\Big(\frac{p}{p-p_0}\Big)^{\frac{1}{p}}\Big(2\,\frac{3^n p_0}{p_0-1}A_0\Big)^{\frac{p_0}{p}} A_1^{1-\frac{p_0}{p}}\,\big\|f\big\|_{L^p}$$

for all $f\in\mathscr{F}$, where $p_0<p<\infty$. Consider now a function h in $\mathscr{F}$. Then $h\in L^{p_0}(\mathbf{R}^n)$ and by assumption $T(h)\in L^{p_0,\infty}(\mathbf{R}^n)$. This gives that $M_d(T(h))\in L^{p_0,\infty}(\mathbf{R}^n)$ by Theorem 6.3.3 (d); then Proposition 6.5.1 [in particular (6.5.1)] is applicable and gives

$$\big\|T(h)\big\|_{L^p} \le 2^{n+p+2+\frac{1}{p}}\big\|M_c^\#(T(h))\big\|_{L^p} \le C_{n,p,p_0}A_0^{\frac{p_0}{p}}A_1^{1-\frac{p_0}{p}}\big\|h\big\|_{L^p}.$$

From this and the density of $\mathscr{F}$ in L^p we derive (6.5.3) for all $f\in L^p(\mathbf{R}^n)$.
~~**Case II:** $p_0=1$. We fix a $\delta\in(0,1)$ and we define the following version of S~~
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$$S_\delta(f) = M_c^\#(|T(f)|^\delta)^{\frac{1}{\delta}}$$

on L^1+L^∞. We prove that S_δ maps L^∞_{fin} to L^∞. Indeed, for $f\in L^\infty_{\text{fin}}$ one has

$$\begin{aligned}
\big\|S_\delta(f)\big\|_{L^\infty} &= \big\|M_c^\#(|T(f)|^\delta)^{\frac{1}{\delta}}\big\|_{L^\infty}\\
&= \big\|M_c^\#(|T(f)|^\delta)\big\|_{L^\infty}^{\frac{1}{\delta}}\\
&= \big\||T(f)|^\delta\big\|_{BMO}^{\frac{1}{\delta}}\\
&\le 2^{\frac{1}{\delta}}\big\|T(f)\big\|_{BMO} && \text{(by Exercise 6.1.7)}\\
&\le 2^{\frac{1}{\delta}}A_1\big\|f\big\|_{L^\infty}.
\end{aligned}$$

At the other endpoint for $f\in L^1$ we write

$$\begin{aligned}
\big\|S_\delta(f)\big\|_{L^{1,\infty}} &= \big\|M_c^\#(|T(f)|^\delta)^{\frac{1}{\delta}}\big\|_{L^{1,\infty}}\\
&= \big\|M_c^\#(|T(f)|^\delta)\big\|_{L^{\frac{1}{\delta},\infty}}^{\frac{1}{\delta}}\\
&\le \Big(\frac{3^{n\delta}\frac{2}{\delta}}{\frac{1}{\delta}-1}\Big)^{\frac{1}{\delta}}\big\||T(f)|^\delta\big\|_{L^{\frac{1}{\delta},\infty}}^{\frac{1}{\delta}} && \text{[by (1.4.7)]}\\
&= \Big(\frac{3^{n\delta}2}{1-\delta}\Big)^{\frac{1}{\delta}}\big\|T(f)\big\|_{L^{1,\infty}}\\
&\le \Big(\frac{3^{n\delta}2}{1-\delta}\Big)^{\frac{1}{\delta}}A_0\big\|f\big\|_{L^1}.
\end{aligned}$$

We interpolate between the estimates $S_\delta: L^\infty_{\text{fin}}\to L^\infty$ and $S_\delta: L^1\to L^{1,\infty}$, applying Theorem 1.3.4 with $\mathscr{F}$, the space of all finitely simple functions on $\mathbf{R}^n$. We deduce that for all $1<p<\infty$ and all $f\in\mathscr{F}$ one has

$$\begin{aligned}\left\|M_c^{\#}(|T(f)|^{\delta})\right\|_{L^{p/\delta}}^{\frac{1}{\delta}} &= \left\|S_\delta(f)\right\|_{L^p} \\ &\le 2\Big(\frac{p}{p-1}\Big)^{\frac{1}{p}}\left[\Big(\frac{3^{n\delta}2}{1-\delta}\Big)^{\frac{1}{\delta}}A_0\right]^{\frac{1}{p}}\big(2^{\frac{1}{\delta}}A_1\big)^{1-\frac{1}{p}}\left\|f\right\|_{L^p}.\end{aligned} \tag{6.5.4}$$

In order to prove (6.5.3) we fix p with $1<p<\infty$ and consider a function h in $\mathscr{F}$. Then $h\in L^1(\mathbf{R}^n)$ and thus by assumption $T(h)$ lies in $L^{1,\infty}(\mathbf{R}^n)$. It follows that $|T(h)|^{\delta}$ lies in $L^{1/\delta,\infty}(\mathbf{R}^n)$ hence $M_d(|T(h)|^{\delta})\in L^{1/\delta,\infty}(\mathbf{R}^n)$ by Theorem 6.3.3 (d). Finally Proposition 6.5.1 applies (with $p_0=1/\delta$) and gives

$$\left\|T(h)\right\|_{L^p}=\Big(\left\||T(h)|^{\delta}\right\|_{L^{p/\delta}}\Big)^{\frac{1}{\delta}}\le\Big(2^{n+\frac{p}{\delta}+2+\frac{\delta}{p}}\left\|M_c^{\#}(|T(h)|^{\delta})\right\|_{L^{p/\delta}}\Big)^{\frac{1}{\delta}}. \tag{6.5.5}$$

Inserting the estimate in (6.5.4) to (6.5.5) we obtain (6.5.3) with h in place of f, where $C_{n,p,1}$ also depends on δ (but we could take $\delta=1/2$). Finally, by the density of $\mathscr{F}$ in $L^p(\mathbf{R}^n)$ we deduce (6.5.3) for all $f\in L^p(\mathbf{R}^n)$. DELETE GREEN TEXT □

Remark 6.5.3. Theorem 6.5.2 is also valid if BMO is replaced by the bigger space BMO_d. In fact one replaces $M_c^{\#}$ by $M_d^{\#}$ and uses a version of Exercise 6.1.7 with BMO_d in place of BMO.

Next, we turn to an application of Theorem 6.5.2.

Theorem 6.5.4. *Let K be a locally integrable function on $\mathbf{R}^n\setminus\{0\}$ which satisfies* (3.3.3) *and* (3.3.4) *(with constant A_2), and associated with K there is a $W\in\mathscr{S}'(\mathbf{R}^n)$ and a sequence $\delta_k\downarrow 0$ as in* (3.3.7)*. Assume that the operator T given by convolution with W admits a bounded extension from L^2 to L^2. Then for all $f\in L^\infty_{\text{fin}}$ we have*

$$\left\|T(f)\right\|_{BMO}\le 2\big(A_2+(2\sqrt{n})^{\frac{n}{2}}\left\|T\right\|_{L^2\to L^2}\big)\left\|f\right\|_{L^\infty}. \tag{6.5.6}$$

Proof. Let us fix a cube Q centered at a point c_Q and let Q^* be the cube with the same center and with sides parallel to those of Q and side length $\ell(Q^*)=2\sqrt{n}\ell(Q)$, where $\ell(Q)$ is the side length of Q. Given a function f bounded whose support has finite measure we split $f=f_0+f_\infty$, where $f_0=f\chi_{Q^*}$ and $f_\infty=f\chi_{(Q^*)^c}$. For fixed $x\in Q$ we claim that $T(f_\infty)(x)$ is well defined and is equal to

$$T(f_\infty)(x)=\lim_{\delta_k\downarrow 0}\int_{|x-y|\ge\delta_k}K(x-y)f(y)\chi_{(Q^*)^c}(y)dy=\int_{(Q^*)^c}K(x-y)f(y)dy,$$

and the reason is that the last integral is absolutely convergent; indeed, for $y\in(Q^*)^c$ and $x\in Q$ we have $|x-y|\ge\sqrt{n}\,\ell(Q)$, thus

$$|K(x-y)|\le\frac{A_1}{|x-y|^n}\le\frac{A_1}{\big(\sqrt{n}\,\ell(Q)\big)^n}$$

and the function f is bounded and supported in a set of finite measure.

We set

$$C_Q=T(f_\infty)(c_Q).$$