by considering the cases $|x| \le 2R$ and $|x| > 2R$. Use this estimate to prove

$$\frac{1}{|B(0,R)|^2} \int_{B(0,R)} \int_{B(0,R)} \big|\mathcal{I}_s(f)(y) - \mathcal{I}_s(f)(z)\big|\,dy\,dz \le C' \big\|f\big\|_{L^{n/s}(\mathbf{R}^n)}$$

by Hölder's inequality, where C' is independent of f and R. By translation, this estimate is valid over any ball and thus over any cube. Then use Exercise 6.4.4.]

6.4.6. Suppose $f_k, f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$, and $|f_k| \le C|f|$ a.e. for $k = 1, 2, \dots$, where C is a positive constant. Suppose that $f_k \to f$ a.e. as $k \to \infty$. Show that

$$M_c^\#(f) \le \liminf_{k\to\infty} M_c^\#(f_k).$$

6.4.7. (a) Let f, f_k, $k = 1, 2, \dots$ be functions in $L^\infty(\mathbf{R}^n)$ such that $\|f_k - f\|_{L^\infty} \to 0$ as $k \to \infty$. Show that $M_c^\#(f_k) \to M_c^\#(f)$ pointwise everywhere.
(b) Let f, f_k, $k = 1, 2, \dots,$ be functions in $L^1(\mathbf{R}^n)$. Suppose that $f_k \to f$ in $L^1(\mathbf{R}^n)$ and the sequence $\{f_k\}_{k=1}^\infty$ is *equicontinuous* in L^1 in the following sense: For every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|A| < \delta \implies \int_A |f_k|\,dy < \varepsilon \qquad \text{for all } k.$$

Prove that $M_c(f_k) \to M_c(f)$ pointwise everywhere, although it may not be the case that $M_c^\#(f_k) \to M_c^\#(f)$.

6.5 Interpolation Using BMO

In this section we discuss an interpolation theorem in which the space L^∞ is replaced by BMO. The sharp function plays a key role in this result. Before doing so, we state a corollary of Theorem 6.4.6 that is quite useful in this type of interpolation.

Proposition 6.5.1. *Let $0 < p_0 < \infty$. Then for any p with $p_0 < p < \infty$ and for all f in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$ satisfying* (6.4.9) *[in particular if $M_d(f) \in L^{p_0,\infty}$] we have*

$$\begin{aligned}\big\|f\big\|_{L^p(\mathbf{R}^n)} &\le \big\|M_d(f)\big\|_{L^p(\mathbf{R}^n)}\\ &\le 2^{n+p+2+\frac{1}{p}} \big\|M_d^\#(f)\big\|_{L^p(\mathbf{R}^n)}\\ &\le 2^{n+p+2+\frac{1}{p}} \big\|M_c^\#(f)\big\|_{L^p(\mathbf{R}^n)}\\ &\le 2^{n+p+3+\frac{1}{p}} \big\|M_c(f)\big\|_{L^p(\mathbf{R}^n)}.\end{aligned} \tag{6.5.1}$$

Analogously, we have