6.3.2. Prove that every cube of length L in $\mathbf{R}^n$ is contained in the union of 3^n dyadic cubes, each having length less than L.

6.3.3. Let $k \in \mathbf{Z}$. Given a cube Q in $\mathbf{R}^n$ of side length L satisfying $2^{k-2} \le L < 2^{k-1}$, prove that there is a dyadic cube D_Q of side length 2^k and a $\sigma = (\sigma_1, \dots, \sigma_n)$, where $\sigma_j \in \{0, 1/2, -1/2\}$ such that such that $Q \subsetneqq 2^k \sigma + D_Q$.

6.3.4. Prove that $\|M_d\|_{L^1(\mathbf{R}^n) \to L^{1,\infty}(\mathbf{R}^n)} = 1$.

6.3.5. Show that $BMO_d(\mathbf{R}^n)$ is a complete.

6.3.6. Prove that the function $\log|x| \, \chi_{x_1>0} \cdots \chi_{x_n>0}$ lies in $BMO_d(\mathbf{R}^n)$ but not in $BMO(\mathbf{R}^n)$.

6.3.7. Prove that the function $\big|\log|x|\big|^p \log\big(\big|\log|x|\big|\chi_{x>0}+1\big)\chi_{x>0}$ lies in $BMO_d(\mathbf{R})$ when $0 < p < 1$. $\big[$*Hint:* Use Exercise 6.1.8.$\big]$

6.3.8. Given a locally integrable function f on $\mathbf{R}^n$, consider the *dyadic average operator* $E_k(f)(x) = \sum_{Q \in \mathscr{D}_k} f_Q \chi_Q(x)$ and the *martingale maximal function* $G(f) = \sup_{k \in \mathbf{Z}} |E_k(f)|$. Prove the following assertions:

(a) G is of weak type $(1,1)$ with constant at most 1.
(b) $E_N(f) \to f$ a.e. as $N \to \infty$ for all $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$. (Relate this to Exercise 1.5.5.)
(c) $E_N(f) \to 0$ as $N \to -\infty$ for all $f \in L^p(\mathbf{R}^n)$ with $1 \le p < \infty$.

6.4 The Sharp Maximal Function

Recall the Hardy–Littlewood maximal function with respect to cubes:

$$M_c(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy$$

defined for $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$. In this section we introduce a related maximal operator that controls the mean oscillation of a function near any point.

Definition 6.4.1. Given a locally integrable function f on $\mathbf{R}^n$, we define its *sharp maximal function* $M_c^\#(f)$ as

$$M_c^\#(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q \big|f(t) - f_Q\big| \, dt, \tag{6.4.1}$$

where the supremum is taken over all cubes Q in $\mathbf{R}^n$ that contain the given point x.

The sharp maximal function is also related to the space BMO. In fact,

$$BMO(\mathbf{R}^n) = \{f \in L^1_{\mathrm{loc}}(\mathbf{R}^n) : \, M_c^\#(f) \in L^\infty(\mathbf{R}^n)\}$$

and $\|f\|_{BMO} = \|M_c^\#(f)\|_{L^\infty}$.