$$f_N(x) = \begin{cases} N & \text{if } f(x) > N, \\ f(x) & \text{if } |f(x)| \le N, \\ -N & \text{if } f(x) < -N \end{cases}$$

satisfies $\|f_N\|_{BMO} \le \frac{9}{4}\|f\|_{BMO}$. $\big[$*Hint:* Write $f_N = \max(-N, \min(f,N))$.$\big]$

6.1.3. Let $1 \le p \le \infty$. Find functions F in $L^p(\mathbf{R}^n)$ and $G \in BMO(\mathbf{R}^n)$ such that FG does not lie in $L^p(\mathbf{R}^n)$.

6.1.4. Show that for all f in $BMO_{\text{balls}}(\mathbf{R}^n)$ and all $r > 0$ we have

$$\big|f_{rB} - f_B\big| \le 2^n\big(1 + \log_2 \max(r, \tfrac{1}{r})\big)\,\big\|f\big\|_{BMO_{\text{balls}}}.$$

6.1.5. Let $a > 0$ and let $f \in BMO(\mathbf{R}^n)$. Let B and B' be balls in $\mathbf{R}^n$ both of radius r whose centers have distance $a\,r$ (these balls could be overlapping). Prove that

$$\big|f_B - f_{B'}\big| \le 2^{n+1}\log_2(a+2)\,\big\|f\big\|_{BMO_{\text{balls}}}.$$

Also show that $\sup_{B,B' \text{ balls with } |B|=|B'|} |f_B - f_{B'}|/\|f\|_{BMO}$ may be unbounded.
$\big[$*Hint:* Pick $m \in \mathbf{Z}$ such that $2^m \le a+2 < 2^{m+1}$ and let x_0 be the midpoint of the line segment joining the centers of B and B'. Consider the ball $B'' = B(x_0, 2^m r)$ and estimate $|f_B - f_{B''}|$ and $|f_{B'} - f_{B''}|$ via telescoping sums, using (6.1.10).$\big]$

6.1.6. Let $f \in BMO(\mathbf{R}^n)$ and $N \in \mathbf{Z}^+$. Verify the following assertions:
(a) For any two cubes Q and Q' of side length 1 contained in $[0, 2^N]^n$ we have

$$\big|f_Q - f_{Q'}\big| \le N2^{n+1}\big\|f\big\|_{BMO}.$$

(b) Let $-\infty < l < L < \infty$. Conclude that for any two cubes Q, Q' of side length 2^l both contained in a cube of side length 2^L the following estimate is valid:

$$\big|f_Q - f_{Q'}\big| \le (L - l + 1)2^{n+1}\big\|f\big\|_{BMO}.$$

$\big[$*Hint:* Part (a). For any interval I of length 1 contained in $[0, 2^N]$ there is a sequence of intervals $I = I_0 \subset I_1 \subset \cdots \subset I_{N-1} \subset [0, 2^N]$ with $|I_j| = 2^j$. Then use (6.1.10).$\big]$

6.1.7. Let Φ be a concave strictly increasing function from $[0,\infty)$ to $[0,\infty)$ that satisfies $\Phi(0) = 0$, $\lim_{t\to\infty}\Phi(t) = \infty$, and $\Phi(t+s) \le \Phi(t) + \Phi(s)$ for all $t, s \ge 0$. Prove that if $f \in BMO$, then $\Phi(|f|)$ lies also in BMO and

$$\big\|\Phi(|f|)\big\|_{BMO} \le 2\Phi(\|f\|_{BMO}).$$

Let $0 < p < 1$. Three important examples of such functions Φ are

$$\Phi(t) = t^p, \quad \Phi(t) = \log(t+1), \quad \Phi(t) = [\log(t+1)]^p.$$