Example 6.1.5. We show that the unbounded function $\log|x|$ lies in $BMO(\mathbf{R}^n)$. Hence $L^\infty(\mathbf{R}^n)$ is a proper subspace of $BMO(\mathbf{R}^n)$.

Indeed, we will show that $\log|x|$ lies in $BMO_{\text{balls}}(\mathbf{R}^n)$. Let $B(x_0,R)$ be a ball. If $|x_0| > 2R$, then for $|x-x_0| \le R$ we have $\frac{1}{2}|x_0| \le |x| \le \frac{3}{2}|x_0|$, hence

$$\frac{1}{v_n R^n}\int\limits_{|x-x_0|\le R} \big|\log|x| - \log|x_0|\big|\,dx = \frac{1}{v_n R^n}\int\limits_{|x-x_0|\le R} \Big|\log\frac{|x|}{|x_0|}\Big|\,dx$$
$$\le \max\Big(\log\frac{3}{2}, \Big|\log\frac{1}{2}\Big|\Big) = \log 2\,.$$

Also, if $|x_0| \le 2R$, then

$$\frac{1}{v_n R^n}\int\limits_{|x-x_0|\le R} \big|\log|x| - \log R\big|\,dx = \frac{1}{v_n R^n}\int\limits_{|x-x_0|\le R} \Big|\log\frac{|x|}{R}\Big|\,dx$$
$$\le \frac{1}{v_n R^n}\int\limits_{|x|\le 3R} \Big|\log\frac{|x|}{R}\Big|\,dx$$
$$= \frac{1}{v_n}\int\limits_{|x|\le 3} \big|\log|x|\big|\,dx = \frac{3^n(n\log 3 - 1) + 2}{n}.$$

We apply Proposition 6.1.3 with $C_{B(x_0,R)}$ being $\log R$ or $\log|x_0|$ to deduce that $\log|x|$ lies in $BMO_{\text{balls}}(\mathbf{R}^n)$ and hence in $BMO(\mathbf{R}^n)$.

The function $\log|x|$ turns out to be a typical element of BMO, but we will make this statement a bit more precise in the next section. It is interesting, however, to notice that BMO does not remain invariant under abrupt cutoffs.

Example 6.1.6. The function $h(x) = \chi_{x>0}\log\frac{1}{x}$ is not in $BMO(\mathbf{R})$. Indeed, the problem is at the origin. Consider the intervals $(-\varepsilon,\varepsilon)$, where $0<\varepsilon<\frac{1}{2}$. We have that

$$h_{(-\varepsilon,\varepsilon)} = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{+\varepsilon} h(x)\,dx = \frac{1}{2\varepsilon}\int_0^{\varepsilon}\log\frac{1}{x}\,dx = \frac{1+\log\frac{1}{\varepsilon}}{2}.$$

But then

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{+\varepsilon}\big|h(x) - h_{(-\varepsilon,\varepsilon)}\big|\,dx \ge \frac{1}{2\varepsilon}\int_{-\varepsilon}^{0}\big|h_{(-\varepsilon,\varepsilon)}\big|\,dx = \frac{1+\log\frac{1}{\varepsilon}}{4}\,,$$

and the latter is clearly unbounded as $\varepsilon\to 0$. This discussion also reveals examples of two BMO functions whose product is not in BMO ($\chi_{(0,\infty)}$ and $\log\frac{1}{|x|}$).

Proposition 6.1.7. *Under the identification of functions whose difference is a constant a.e., BMO is a complete normed linear space, i.e., a Banach space.*

Proof. Let $\{f_k\}_{k=1}^\infty$ be a Cauchy sequence in BMO. Then, given $\varepsilon>0$ there is a $k_0\in\mathbf{Z}^+$ such that for any cube Q and any $k,m\ge k_0$ we have

$$\frac{1}{|Q|}\int_Q |f_k - (f_k)_Q - (f_m - (f_m)_Q)|\,dx \le \big\|f_k - f_m\big\|_{BMO} < \varepsilon. \tag{6.1.9}$$

Thus $\{(f_k - (f_k)_Q)\chi_Q\}_{k=1}^\infty$ is a Cauchy sequence in L^1 and thus it converges in L^1 to a function F^Q supported in Q with $\int_{\mathbf{R}^n} F^Q\,dx = 0$. Note that if $Q \subsetneqq Q'$ then $(f_k)_Q - (f_k)_{Q'}$ converges to a constant as $k \to \infty$, thus $F^{Q'} - F^Q$ equals a constant $C(Q', Q)$ on Q. Moreover, $(F^Q)_Q = 0$ for any cube Q.

Let $Q_N = [-N,N]^n$, $N = 1,2,\dots$. We define a function F by setting $F = F^{Q_1}$ on Q_1 and $F = F^{Q_N} - C(Q_N, Q_{N-1})$ on Q_N for $N \ge 2$. Clearly F is well defined and lies in $L^1_{\text{loc}}(\mathbf{R}^n)$. The crucial observation is that for every cube Q there is a constant c_Q such that $F = F^Q + c_Q$ on Q. To see this, given a cube Q we find the least $N \ge 2$ such that $Q \subsetneqq Q_N$. Then if $c_Q = C(Q_N, Q) - C(Q_N, Q_{N-1})$ we have

$$F = F^{Q_N} - C(Q_N,Q_{N-1}) = F^Q + C(Q_N,Q) - C(Q_N,Q_{N-1}) = F^Q + c_Q \qquad \text{on } Q.$$

Note that since $(F^Q)_Q = 0$, we have $F_Q = c_Q$. Letting $m \to \infty$ in (6.1.9) yields

$$\frac{1}{|Q|}\int_Q |f_k - (f_k)_Q - F^Q|\,dx \le \varepsilon \qquad \text{for for } k \ge k_0 \text{ and any cube } Q.$$

But as $F^Q = F - c_Q = F - F_Q$ on Q we have

$$\sup_Q \frac{1}{|Q|}\int_Q |f_k - (f_k)_Q - (F - F_Q)|\,dx \le \varepsilon \qquad \text{for any } k \ge k_0.$$

Thus $F \in BMO$ as $\|F\|_{BMO} \le \|f_{k_0}\|_{BMO} + \varepsilon < \infty$, and $f_k \to F$ in BMO as $k \to \infty$. □

We now examine some basic properties of BMO functions. For a ball B and $a > 0$, we denote by aB the ball that is concentric with B and whose radius is a times the radius of B.

Proposition 6.1.8. *Let f be in $BMO_{\text{balls}}(\mathbf{R}^n)$ and let B and B' be balls in $\mathbf{R}^n$.*
(i) *If $B \subset B'$, then*

$$\big|f_B - f_{B'}\big| \le \frac{|B'|}{|B|}\big\|f\big\|_{BMO_{\text{balls}}}. \tag{6.1.10}$$

(ii) *Let $B = B_0 \subset B_1 \subset \cdots \subset B_m = B'$, where B_i is a ball of radius at most twice that of the ball B_{i-1} for each $i = 1,\dots,m$. Then we have*

$$\big|f_B - f_{B'}\big| \le 2^n m\big\|f\big\|_{BMO_{\text{balls}}}. \tag{6.1.11}$$

(iii) *For any $\delta > 0$ there is a constant $C_{n,\delta}$ such that if B is centered at $x_0 \in \mathbf{R}^n$ and has radius R, then we have*

$$R^\delta \int_{\mathbf{R}^n} \frac{\big|f(x) - f_B\big|}{(R + |x - x_0|)^{n+\delta}}\,dx \le C_{n,\delta}\big\|f\big\|_{BMO_{\text{balls}}}. \tag{6.1.12}$$

An analogous estimate holds for cubes with center x_0 and side length R.