to be the least index $s > j_2$ such that $B_s$ is disjoint from $B_{j_1} \cup B_{j_2}$. This means that for all indices $m$ (if any) with $j_2 < m < j_3$, $B_m$ intersects $B_{j_1}$ or $B_{j_2}$.

Having chosen $j_1, j_2, \dots, j_i$, let $j_{i+1}$ be the least index $s > j_i$ such that $B_s$ is disjoint from $B_{j_1} \cup \cdots \cup B_{j_i}$. As before the selection procedure ensures that if the index $m$ satisfies $j_i < m < j_{i+1}$, then $B_m$ intersects one of the balls $B_{j_1}, \dots, B_{j_i}$. Since we have a finite number of balls, this process will terminate when an index $j_L$ is found such that for all $m > j_L$ the balls $B_m$ intersect one of the $B_{j_1}, \dots, B_{j_L}$.

We have now selected pairwise disjoint balls $B_{j_1}, \dots, B_{j_L}$. If some $B_m$ was not selected, that is, $m \notin \{j_1, \dots, j_L\}$, as observed before, then $B_m$ must intersect a selected ball $B_{j_r}$ for some $j_r < m$. Then $B_m$ has smaller size than $B_{j_r}$ and we must have $B_m \subset 3B_{j_r}$. This shows that the union of the unselected balls is contained in the union of the triples of the selected balls. Therefore, the union of all balls is contained in the union of the triples of the selected balls. Thus

$$\left| \bigcup_{i=1}^{N} B_i \right| \le \left| \bigcup_{r=1}^{L} 3B_{j_r} \right| \le \sum_{r=1}^{L} |3B_{j_r}| = 3^n \sum_{r=1}^{L} |B_{j_r}|,$$

and the required conclusion follows. $\square$

It was noted earlier that $\mathcal{M}(f)$ and $M(f)$ are never in $L^1$ if $f$ is not zero a.e. However, it is true that these functions lie in $L^{1,\infty}$ when $f$ is in $L^1$.

**Theorem 1.4.6.** *For any measurable function $f$ and any $\lambda > 0$ we have*

$$\big|\{M(f) > \lambda\}\big| \le \frac{3^n}{\lambda} \int_{\{M(f)>\lambda\}} |f(y)|\, dy. \tag{1.4.3}$$

*Consequently, the uncentered Hardy–Littlewood maximal operator $M$ maps $L^1(\mathbf{R}^n)$ to $L^{1,\infty}(\mathbf{R}^n)$ with constant at most $3^n$.*

*Proof.* We claim that the set $E_\lambda = \{x \in \mathbf{R}^n : M(f)(x) > \lambda\}$ is open. Indeed, for $x \in E_\lambda$, there is an open ball $B_x$ that contains $x$ such that $|B_x|^{-1} \int_{B_x} |f(y)|\, dy > \lambda$. Then $M(f)(y) > \lambda$ for any $y \in B_x$, and thus $B_x \subseteq E_\lambda$. This proves that $E_\lambda$ is open.

Let $K$ be a compact subset of $E_\lambda$. For each $x \in K$ there exists an open ball $B_x$ containing the point $x$ such that

$$\int_{B_x} |f(y)|\, dy > \lambda |B_x|. \tag{1.4.4}$$

Observe that $B_x \subset E_\lambda$ for all $x$. By compactness there exists a finite subcover $\{B_{x_1}, \dots, B_{x_N}\}$ of $K$. Using Lemma 1.4.5 we find a subcollection of pairwise disjoint balls $B_{x_{j_1}}, \dots, B_{x_{j_L}}$ such that (1.4.2) holds. Using (1.4.4) and (1.4.2) we obtain

$$|K| \le \left| \bigcup_{i=1}^{N} B_{x_i} \right| \le 3^n \sum_{i=1}^{L} |B_{x_{j_i}}| \le \frac{3^n}{\lambda} \sum_{i=1}^{L} \int_{B_{x_{j_i}}} |f(y)|\, dy \le \frac{3^n}{\lambda} \int_{E_\lambda} |f(y)|\, dy,$$