We now show that the seminorms $\|f\|_{BMO_{\text{balls}}}$ and $\|f\|_{BMO}$ are in fact comparable; thus the spaces $BMO(\mathbf{R}^n)$ and $BMO_{\text{balls}}(\mathbf{R}^n)$ contain the same functions.

Given any cube Q in $\mathbf{R}^n$, we let B be the smallest ball that contains it. Let v_n be the volume of the unit ball. Then

$$\frac{1}{|Q|}\int_Q \big|f(x)-f_B\big|\,dx \le \frac{|B|}{|Q|}\frac{1}{|B|}\int_B \big|f(x)-f_B\big|\,dx \le \frac{v_n\sqrt{n^n}}{2^n}\,\big\|f\big\|_{BMO_{\text{balls}}}.$$

It follows from Proposition 6.1.3 that $\|f\|_{BMO} \le 2^{1-n}v_n\sqrt{n^n}\,\|f\|_{BMO_{\text{balls}}}$. Now given a ball B find the smallest cube Q that contains it. Then write

$$\frac{1}{|B|}\int_B \big|f(x)-f_Q\big|\,dx \le \frac{|Q|}{|B|}\frac{1}{|Q|}\int_Q \big|f(x)-f_Q\big|\,dx \le \frac{2^n}{v_n}\,\big\|f\big\|_{BMO},$$

and this implies $\|f\|_{BMO_{\text{balls}}} \le 2^{n+1}v_n^{-1}\,\|f\|_{BMO}$. We conclude that the spaces BMO and BMO_{balls} have comparable seminorms, hence they are isomorphic.

Proposition 6.1.4. *If $f \in BMO$, then $|f| \in BMO$. f,g are real-valued BMO functions, then so are* $\max(f,g)$ *and* $\min(f,g)$*. Moreover,*

$$\big\||f|\big\|_{BMO} \le 2\big\|f\big\|_{BMO}, \tag{6.1.5}$$

$$\big\|\max(f,g)\big\|_{BMO} \le \frac{3}{2}\big\|f\big\|_{BMO}+\frac{3}{2}\big\|g\big\|_{BMO}, \tag{6.1.6}$$

$$\big\|\min(f,g)\big\|_{BMO} \le \frac{3}{2}\big\|f\big\|_{BMO}+\frac{3}{2}\big\|g\big\|_{BMO}. \tag{6.1.7}$$

Proof. To prove (6.1.5), note that for each cube Q we have $\big||f|-|f_Q|\big| \le |f-f_Q|$, which implies

$$\big||f|-|f_Q|\big|_Q \le |f-f_Q|_Q \le \big\|f\big\|_{BMO}. \tag{6.1.8}$$

Thus, for each cube Q there is a constant $C_Q = |f_Q|$ such that (6.1.8) holds. Appealing to Proposition 6.1.3 we deduce (6.1.5). Next, note that

$$\max(f,g) = \frac{f+g+|f-g|}{2} \qquad \text{and} \qquad \min(f,g) = \frac{f+g-|f-g|}{2}.$$

Then we obtain the estimate

$$\begin{aligned}\big\|\max(f,g)\big\|_{BMO} &\le \frac{\big\|f\big\|_{BMO}+\big\|g\big\|_{BMO}+\big\||f-g|\big\|_{BMO}}{2}\\ &\le \frac{\big\|f\big\|_{BMO}+\big\|g\big\|_{BMO}+2\big\|f-g\big\|_{BMO}}{2}\\ &\le \frac{\big\|f\big\|_{BMO}+\big\|g\big\|_{BMO}+2\big\|f\big\|_{BMO}+2\big\|g\big\|_{BMO}}{2},\end{aligned}$$

from which we obtain (6.1.6). Likewise we obtain (6.1.7). □