5.4.3. Let $s>0$ and $1<p<\infty$. Let $\phi \in \mathscr{C}_0^\infty(\mathbf{R}^n)$ be equal to 1 on the unit ball and vanishing outside the double of the unit ball. Let $\phi^\varepsilon(x)=\phi(\varepsilon x)$. Prove that for any $f \in \mathscr{S}(\mathbf{R}^n)$, the sequence $f\phi^\varepsilon$ converges to f in $L_s^p(\mathbf{R}^n)$. Then use Theorem 5.4.8 to conclude that smooth functions with compact support are dense in $L_s^p(\mathbf{R}^n)$. $\big[$*Hint:* Show that $\{f\phi^\varepsilon\}_{\varepsilon>0}$ is Cauchy in $L_s^p(\mathbf{R}^n)$ and converges to f in $L^p(\mathbf{R}^n)$ as $\varepsilon \to 0.\big]$

5.4.4. Let $1<p<n$, $\frac{n}{n-1}<q<\infty$ and $\frac{1}{p}-\frac{1}{q}=\frac{1}{n}$. Suppose that $f,\Delta f$ lie in $L^p(\mathbf{R}^n)$. Prove that f lies in $L_1^q(\mathbf{R}^n)$.

5.4.5. Let $1<p<\infty$ and $s\geq 0$.
(a) Let $f\in L_s^p(\mathbf{R}^n)$ and $g\in L^1(\mathbf{R}^n)$. Prove that $f*g$ lies in $L_s^p(\mathbf{R}^n)$.
(b) Let $f\in L_s^p(\mathbf{R}^n)$ and let $g\in L^r(\mathbf{R}^n)$ for some r in $(1,\infty)$. Prove that $f*g$ lies in $L_s^q(\mathbf{R}^n)$ when $1<q<\infty$ and $1+1/q=1/p+1/r$.

5.4.6. (Fractional integration by parts for Sobolev spaces) Let $s>0$, $f\in L_s^p(\mathbf{R}^n)$, and $g\in L_s^{p'}(\mathbf{R}^n)$, $1<p<\infty$. Show that for any $t\in\mathbf{R}$ we have

$$\int_{\mathbf{R}^n} g\,(I-\Delta)^{\frac{s}{2}+it} f\,dx = \int_{\mathbf{R}^n} f\,(I-\Delta)^{\frac{s}{2}+it} g\,dx.$$

$\big[$*Hint:* Use Exercise 5.2.1, Corollary 5.3.3, and density.$\big]$

5.4.7. Show that translations, dilations, and modulations $M^a f(x)=f(x)e^{2\pi i x\cdot a}$ preserve $L_s^p(\mathbf{R}^n)$ for any $s\in\mathbf{R}$. $\big[$*Hint:* Use the result of Example 5.3.9.$\big]$

5.4.8. Let $\varphi\in\mathscr{S}(\mathbf{R}^n)$ and $f\in L_s^p(\mathbf{R}^n)$ for $1<p<\infty$ and $s\in\mathbf{R}$. Prove that $\varphi f\in L_s^p$. $\big[$*Hint:* Write $(I-\Delta)^{\frac{s}{2}}(\varphi f)(x)$ as

$$\int_{\mathbf{R}^n}\widehat{\varphi}(\eta)e^{2\pi i x\cdot\eta}\left[\int_{\mathbf{R}^n}\left(\frac{1+4\pi^2|\xi+\eta|^2}{1+4\pi^2|\xi|^2}\right)^{\frac{s}{2}}[(I-\Delta)^{\frac{s}{2}}f]\widehat{\ }(\xi)e^{2\pi i x\cdot\xi}d\xi\right]d\eta$$

and then use Example 5.3.9.$\big]$

5.4.9. Let u be the inverse Fourier transform of $t^{-1}(\log t)^{-1}\chi_{t\geq 2}$ on the real line. Show that u lies in $L_s^2(\mathbf{R})$ for all $s\leq 1/2$ but $u\notin L^\infty(\mathbf{R})$. How is this example related to Theorem 5.4.9?

5.4.10. Let $s,t\in\mathbf{R}$. Suppose that $|\partial^\alpha\sigma(\xi)|\leq C_\alpha(1+|\xi|)^{s-t}|\xi|^{-|\alpha|}$ for all multi-indices α and all $\xi\in\mathbf{R}^n\setminus\{0\}$. Prove that the operator $T_\sigma(\varphi)=(\widehat{\varphi}\sigma)^\vee$, initially defined for $\varphi\in\mathscr{S}(\mathbf{R}^n)$, admits a bounded extension from $L_s^p(\mathbf{R}^n)$ to $L_t^p(\mathbf{R}^n)$.

5.5 Interpolation of Analytic Families of Operators

In this section we prove an interpolation result for families of operators indexed by a complex parameter in which they depend analytically. We begin with a lemma that allows us to approximate a general $\mathscr{C}_0^\infty$ function by a family of $\mathscr{C}_0^\infty$ functions which are analytic in an auxiliary variable.