$$\big\|f\big\|_{L^q} \le C'_{s,n}\big\|\mathcal{I}_s\big(|(I-\Delta)^{\frac{s}{2}}f|\big)\big\|_{L^q} \le C''_{s,n}\big\|f\big\|_{L^p_s}.$$

(b) Given q satisfying $\frac{n}{s}=p\le q<\infty$, there is an r in $[1,\infty)$ such that

$$1+\frac{1}{q}=\frac{1}{p}+\frac{1}{r}.$$

Then $1<\frac{s}{n}+\frac{1}{r}$, which implies that $(-n+s)r>-n$. Thus, the function $|x|^{-n+s}\chi_{|x|\le 2}$ is integrable to the rth power, which implies that G_s is in $L^r(\mathbf{R}^n)$. As f in L^p_s can be written $G_s*(I-\Delta)^{\frac{s}{2}}f$, Young's inequality (Proposition 2.4.2) gives that

$$\big\|f\big\|_{L^q(\mathbf{R}^n)} \le \big\|G_s\big\|_{L^r(\mathbf{R}^n)}\big\|(I-\Delta)^{\frac{s}{2}}f\big\|_{L^p(\mathbf{R}^n)} = C_{n,s}\big\|f\big\|_{L^p_s}.$$

(c) Let $s>n/p$ and M be the largest integer strictly less than $s-\frac{n}{p}$. Then for all $|\alpha|\le M$, by Proposition 5.2.3, $\partial^\alpha G_s(x)$ decays exponentially when $|x|\ge 2$ and it is bounded by a constant multiple of $\max\big(|x|^{-n+s-|\alpha|},\log\frac{2}{|x|},1\big)$ when $0<|x|<2$. This function lies in $L^{p'}(B(0,2))$ when $|\alpha|<s-n/p$ and thus $\partial^\alpha G_s$ lies in $L^{p'}(\mathbf{R}^n)$ for all $|\alpha|\le M$. Let $f\in L^p_s(\mathbf{R}^n)$. Applying Theorem 1.7.1 (with $g=(I-\Delta)^{s/2}f$, $\varphi=G_s$, and $m=M$)[4] we obtain that $f=G_s*(I-\Delta)^{s/2}f$ lies in $\mathscr{C}^M(\mathbf{R}^n)$ and satisfies

$$\partial^\alpha f=\partial^\alpha(G_s*(I-\Delta)^{s/2}f)=(\partial^\alpha G_s)*(I-\Delta)^{s/2}f$$

for all $|\alpha|\le M$. The boundedness and uniform continuity of this function are consequences of Theorem 1.6.7. Hölder's inequality now yields

$$\sum_{|\alpha|\le M}\big\|\partial^\alpha f\big\|_{L^\infty} \le \sum_{|\alpha|\le M}\big\|\partial^\alpha G_s\big\|_{L^{p'}}\big\|(I-\Delta)^{s/2}f\big\|_{L^p} = \Big(\sum_{|\alpha|\le M}\big\|\partial^\alpha G_s\big\|_{L^{p'}}\Big)\big\|f\big\|_{L^p_s}$$

and thus the claimed embedding is valid.

In the event that $N=s-\frac{n}{p}$ is an integer, for $p\le q<\infty$ we notice that

$$\big\|f\big\|_{L^q_N} = \big\|(I-\Delta)^{\frac{N}{2}}f\big\|_{L^q} \le C\big\|(I-\Delta)^{\frac{N}{2}}f\big\|_{L^p_{s-N}} = C\big\|f\big\|_{L^p_s},$$

where the preceding inequality is a consequence of the assertion in part (b). □

Exercises

5.4.1. Let $s\in\mathbf{Z}^+$ and $f\in L^p_s(\mathbf{R}^n)$ for $1<p<\infty$. Suppose that g is a $\mathscr{C}^s$ function on $\mathbf{R}^n$ with the property $\partial^\alpha g\in L^\infty(\mathbf{R}^n)$ for all $|\alpha|\le s$. Prove the $fg\in L^p_s(\mathbf{R}^n)$.

5.4.2. Let $\widehat{\Phi}$ be a $\mathscr{C}_0^\infty$ function on $\mathbf{R}^n$ that equals 1 in a neighborhood of the origin. Prove that the function $\xi\mapsto(1+|\xi|^2)^{s/2}|\xi|^{-s}\big(1-\widehat{\Phi}(\xi)\big)$ lies in $\mathscr{M}_p(\mathbf{R}^n)$ for all $1<p<\infty$.

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[4] Strictly speaking, to apply Theorem 1.7.1, we need G_s to be in $\mathscr{C}^M(\mathbf{R}^n)$, but G_s may be discontinuous at the origin. So, we need a version of Theorem 1.7.1 in which the hypothesis φ lies in $\mathscr{C}^M(\mathbf{R}^n)\cap L^{q'}(\mathbf{R}^n)$ is replaced by φ lies in $\mathscr{C}^M(\mathbf{R}^n\setminus\{0\})\cap L^{q'}(\mathbf{R}^n)$. Then the conclusion is that $\varphi*g$ lies in $\mathscr{C}^M(\mathbf{R}^n\setminus\{0\})$ and $\partial^\alpha(\varphi*g)$ is equal to the continuous and bounded function $(\partial^\alpha\varphi)*g$ on $\mathbf{R}^n\setminus\{0\}$ for all $|\alpha|\le m$.