**1.3.4.** Let $(X,\mu)$ and $(Y,\nu)$ be $\sigma$-finite measure spaces. Let $T$ be a subadditive operator defined on $L^1(X,\mu)+L^\infty(X,\mu)$ that takes values in the space of measurable functions on $(Y,\nu)$ and that satisfies $|T(f)| \le T(|f|)$ for all $f \in L^1+L^\infty$. Suppose that $T$ maps $L^1$ to $L^{1,\infty}$ with bound $A_0$ and $L^\infty$ to itself with bound $A_1>0$. Given $1<p<\infty$, prove that $T$ maps $L^p(X)$ to $L^p(Y)$ with norm at most

$$\frac{p}{p-1} A_0^{\frac{1}{p}} A_1^{1-\frac{1}{p}}.$$

$\big[$*Hint:* Given $\lambda>0$, $\gamma\in(0,1)$, and $f$ measurable, write $|f| = f_0+f_1$, where $f_0 = \max\big(|f|-\frac{\gamma\lambda}{A_1},0\big)$ and $f_1 = \min\big(|f|,\frac{\gamma\lambda}{A_1}\big)$. Then $\{T(|f|)>\lambda\} \subsetneqq \{|T(f_0)|>(1-\gamma)\lambda\}$. Then choose a suitable $\gamma$.$\big]$

**1.3.5.** Let $(X,\mu)$, $(Y,\nu)$ be $\sigma$-finite measure spaces, and let $0<p_0<p_1\le\infty$. Define $p$ via $\frac{1-\theta}{p_0}+\frac{\theta}{p_1}=\frac{1}{p}$, where $0<\theta<1$. Let $T$ be a subadditive operator defined on $L^{p_0}(X)+L^{p_1}(X)$ and taking values in the space of measurable functions on $Y$. Suppose $T$ maps $L^{p_0}$ to $L^\infty$ with norm $A_0$ and $L^{p_1}$ to $L^\infty$ with norm $A_1$. Prove that $T$ maps $L^p$ to $L^\infty$ with norm at most $2A_0^{1-\theta}A_1^\theta$.

**1.3.6.** Let $(X,\mu)$ and $(Y,\nu)$ be $\sigma$-finite measure spaces. Let $0<p<p_1\le\infty$, $0<B<\infty$, and let $\Phi:\ [0,\infty)\to[0,\infty)$ be a measurable function such that

$$A=\int_0^1 \lambda^{p-1}\Phi(1/\lambda)\,d\lambda<\infty.$$

Let $T$ be a linear operator that maps $L^{p_1}(X)$ to $L^{p_1,\infty}(Y)$ with norm $B$ that satisfies

$$\nu\big(\{y\in Y:\ |T(f)(y)|>\lambda\}\big)\le A\int_X \Phi\Big(\frac{|f(x)|}{\lambda}\Big)\,d\mu$$

for all finite simple functions $f$ on $X$ and all $\lambda>0$. Prove that $T$ has a bounded extension from $L^p(X)$ to itself. $\big[$*Hint:* Set $f^\lambda = f\chi_{|f|>\lambda}$ and $f_\lambda = f\chi_{|f|\le\lambda}$. When $p_1<\infty$, add the estimates

$$p\lambda^{p-1}\nu(\{|T(f^\lambda)|>\lambda\})\le Ap\lambda^{p-1}\int_{|f|>\lambda}\Phi\Big(\frac{|f(x)|}{\lambda}\Big)d\mu$$

and

$$p\lambda^{p-1}\nu(\{|T(f_\lambda)|>\lambda\})\le p\lambda^{p-1}B^{p_1}\int_{|f|\le\lambda}\frac{|f(x)|^{p_1}}{\lambda^{p_1}}d\mu\,,$$

and integrate over $\lambda$ to estimate $\frac{1}{2^p}\|T(f)\|_{L^p}^p$. In the case where $p_1=\infty$, use

$$\nu(\{|T(f)|>2B\lambda\})\le\nu(\{|T(f^\lambda)|>B\lambda\})$$

to complete the proof.$\big]$

**1.3.7. (Vector-valued Marcinkiewicz interpolation)** Let $(X,\mu)$, $(Y,\nu)$ be $\sigma$-finite measure spaces and $0<p_0<p_1\le\infty$. Fix quasi-normed spaces $Z,W$. Define