5.1.8. Prove that $(-\Delta)^{z/2}\varphi$ is a bounded function whenever
(a) $\mathrm{Re}\, z > -n$ and $\varphi \in \mathscr{S}(\mathbf{R}^n)$; or
(b) $z \in \mathbf{C}$ and the Fourier transform of $\varphi \in \mathscr{S}(\mathbf{R}^n)$ vanishes in a neighborhood of 0.

5.1.9. Fix $w \in \mathbf{C}$ with $\mathrm{Re}\, w > 0$. Show that for every $\varphi \in \mathscr{C}_0^\infty(\mathbf{R}^n)$ there is a constant $C(n,w,\varphi)$ such that

$$\big|(-\Delta)^{w/2}\varphi(x)\big| \le C(n,w,\varphi)(1+|x|)^{-n-\mathrm{Re}\, w}.$$

Moreover, prove that for any $s > 0$ and $s \notin 2\mathbf{Z}$ and every nonnegative and nonzero $\varphi \in \mathscr{C}_0^\infty(\mathbf{R}^n)$, there are constants $C, K > 0$ such that for all $|x| > K$ one has

$$\big|(-\Delta)^{s/2}\varphi(x)\big| \ge C\,|x|^{-n-s}.$$

$\big[$*Hint:* Identity (5.1.3) applied to φ can be extended to complex numbers z with $\mathrm{Re}\, z < 0$ by analytic continuation, for large values of x.$\big]$

5.2 Bessel Potentials

In this section we study an *adjustment* of the Riesz potentials in which we replace $-\Delta$ by $I - \Delta$, where I is the identity operator. This simple modification allows one to define the action of $(I-\Delta)^{z/2}$ on $\mathscr{S}(\mathbf{R}^n)$ for all $z \in \mathbf{C}$. Notice that $(1+4\pi^2|\cdot|^2)^{z/2}\widehat{\varphi}$ lies in $\mathscr{S}(\mathbf{R}^n)$, whenever $\varphi \in \mathscr{S}(\mathbf{R}^n)$, thus so does $(I-\Delta)^{z/2}\varphi$, by Proposition 2.6.12. In fact, $(I-\Delta)^{z/2}$ is a one-to-one and onto mapping from $\mathscr{S}(\mathbf{R}^n)$ to $\mathscr{S}(\mathbf{R}^n)$.

Definition 5.2.1. Let z be a complex number satisfying $0 < \mathrm{Re}\, z < \infty$. The *Bessel potential* of order z is the operator

$$\mathcal{J}_z(f) = (I-\Delta)^{-\frac{z}{2}}(f) = \big((1+4\pi^2|\cdot|^2)^{-\frac{z}{2}}\widehat{f}\,\big)^\vee,$$

initially acting on Schwartz functions f.

We denote by G_z the kernel of $\mathcal{J}_z$, i.e.,

$$G_z = \big((1+4\pi^2|\cdot|^2)^{-\frac{z}{2}}\big)^\vee,$$

which a priori is a tempered distribution. The Bessel potential is the operator

$$\mathcal{J}_z(\varphi) = \varphi * G_z\,, \qquad \varphi \in \mathscr{S}(\mathbf{R}^n).$$

The Bessel potential is obtained by replacing $4\pi^2|\xi|^2$ in the Riesz potential by the smooth term $1+4\pi^2|\xi|^2$. This adjustment smooths the function near zero, and this translates into rapid decay for its inverse Fourier transform at infinity.

The next result quantifies the behavior of G_s near zero and near infinity for $s > 0$. To describe this behavior we introduce a function H_s on $\mathbf{R}^n \setminus \{0\}$ by setting