$$\|\varphi\|_{L^2}^2 \le \|\mathcal{I}_z(\varphi)\|_{L^q} \|(-\Delta)^{\bar{z}/2}\varphi\|_{L^{q'}}.$$

5.1.3. Show that for any $t > 0$ there is a constant $c_{n,t}$ such that

$$\|f\|_{L^1(\mathbf{R}^n)} \le c_{n,t} \|(-\Delta)^{\frac{n+t}{2}} f\|_{L^2(\mathbf{R}^n)}^{\frac{n}{n+t}} \|f\|_{L^2(\mathbf{R}^n)}^{\frac{t}{n+t}}$$

is valid for all $f \in L^2(\mathbf{R}^n)$. Note that the Fourier transform of $(-\Delta)^{\frac{n+t}{2}} f$ is well defined when $f \in L^2$ and so does its L^2 norm, which could be infinite. $\big[$*Hint:* First prove that $\int_{\mathbf{R}^n} |f(x)|\,dx \le C\big(\int_{\mathbf{R}^n} |f(x)|^2(|x|^{n+t}+1)\,dx\big)^{1/2}$, then apply this inequality to $f(\lambda x)$, and optimize over λ.$\big]$

5.1.4. Let s be an even integer with $0 < s < n$, and let $1 < p < \frac{n}{s}$ and $\frac{n}{n-s} < q < \infty$ be related as in (5.1.5). Suppose that $(-\Delta)^{\frac{s}{2}} f \in L^p(\mathbf{R}^n)$ for a given $f \in \mathscr{S}'(\mathbf{R}^n)$. Prove that f coincides with an L^q function whose norm satisfies the estimate

$$\|f\|_{L^q} \le C(n,p,s) \|(-\Delta)^{\frac{s}{2}} f\|_{L^p},$$

where $C(n,p,s) < \infty$. $\big[$*Hint:* Use Theorem 5.1.3 and that $\mathcal{I}_s$ is self-adjoint.$\big]$

5.1.5. Let f be a tempered and locally integrable function on $\mathbf{R}^n$. Suppose that either (a) $n \ge 2$ and the distributional derivatives $\partial_j f$ lie in $L^{p_1}(\mathbf{R}^n) \cap L^{p_2}(\mathbf{R}^n)$ for all $j = 1, \dots, n$, where $1 < p_1 < n < p_2 < \infty$; or (b) $n \ge 3$ and the distributional Laplacian Δf lies in $L^{p_1}(\mathbf{R}^n) \cap L^{p_2}(\mathbf{R}^n)$, where $1 \le p_1 < \frac{n}{2} < p_2 \le \infty$. Prove that f lies in $L^\infty(\mathbf{R}^n)$.
$\big[$*Hint:* (a) Use the identity $f = \sum_{j=1}^n \mathcal{I}_1(R_j(\partial_j f))$. (b) Write $f = -\mathcal{I}_2(\Delta f)$.$\big]$

5.1.6. For $0 < s < n$ define the *fractional maximal function*

$$M^s(f)(x) = \sup_{t>0} \frac{1}{(v_n t^n)^{\frac{n-s}{n}}} \int_{|y|\le t} |f(x-y)|\,dy, \qquad f \in L^1_{\mathrm{loc}}(\mathbf{R}^n),$$

where $v_n = |B(0,1)|$. Show that for some finite constant $C(n,s)$ we have

$$M^s(f) \le C(n,s)\mathcal{I}_s(|f|).$$

5.1.7. For continuous functions f on $\mathbf{R}^n$ define the *difference operator* $D_h f(x) = f(x+h) - f(x)$ for $x, h \in \mathbf{R}^n$. Let $0 < s < m$ and $m \in \mathbf{Z}^+$. Prove that for $f \in \mathscr{S}(\mathbf{R}^n)$ we have

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \big|\overbrace{D_t \circ \cdots \circ D_t}^{m \text{ times}} f(x)\big|^2 dx \frac{dt}{|t|^{n+2s}} = C(m,n,s)\|(|\cdot|^s \widehat{f})^\vee\|_{L^2}^2,$$

where $C(m,n,s) = \displaystyle\int_{\mathbf{R}^n} |e^{2\pi i t_1} - 1|^{2m} |t|^{-n-2s} dt < \infty$.
$\big[$*Hint:* Use that $(D_t \circ \cdots \circ D_t f)\widehat{\ }(\xi) = \widehat{f}(\xi)(e^{2\pi i \xi\cdot t} - 1)^m$.$\big]$