We are left with the case $p=1$ when $q=\frac{n}{n-s}$. The result in this case also follows from (5.1.16) by the weak-type $(1,1)$ property of M (Theorem 1.4.6). Indeed for all $\lambda>0$ we have

$$\begin{aligned}\Big|\Big\{C_{n,s,1}M(f)^{\frac{n-s}{n}}\big\|f\big\|_{L^1}^{\frac{s}{n}}>\lambda\Big\}\Big| &= \Big|\Big\{M(f)>\Big(\frac{\lambda}{C_{n,s,1}\big\|f\big\|_{L^1}^{\frac{s}{n}}}\Big)^{\frac{n}{n-s}}\Big\}\Big|\\ &\le 3^n\Big(\frac{C_{n,s,1}\big\|f\big\|_{L^1}^{\frac{s}{n}}}{\lambda}\Big)^{\frac{n}{n-s}}\big\|f\big\|_{L^1}\\ &= \Big(C(n,s)\frac{\big\|f\big\|_{L^1}}{\lambda}\Big)^{\frac{n}{n-s}}.\end{aligned}$$

This is the claimed estimate (5.1.12). □

Example 5.1.4. It follows from (5.1.7) that $\mathcal{I}_s$ does not map $L^1(\mathbf{R}^n)$ to $L^{\frac{n}{n-s}}(\mathbf{R}^n)$.

Additionally, $\mathcal{I}_s$ does not map $L^{\frac{n}{s}}(\mathbf{R}^n)$ to $L^\infty(\mathbf{R}^n)$. To see this, let $0<s<n$ and pick δ such that $0<\delta<\frac{n-s}{s}$. Consider the function $h(x)=|x|^{-s}(\log\frac{1}{|x|})^{-\frac{s}{n}(1+\delta)}$ for $|x|\le 1/e$ and zero otherwise. A straightforward substitution based on the convergence of the integral $\int_0^{1/e}r^{-1}[\log(r^{-1})]^{-(1+\delta)}dr$ indicates that h lies in $L^{\frac{n}{s}}(\mathbf{R}^n)$. On the other hand for $|x|\le\frac{1}{100}$ we have

$$\begin{aligned}\mathcal{I}_s(h)(x) &= c\int_{|y|\le 1/e}|y|^{-s}\Big(\log\frac{1}{|y|}\Big)^{-\frac{s}{n}(1+\delta)}|x-y|^{s-n}\,dy\\ &\ge c\int_{2|x|\le|y|\le 1/e}|y|^{-s}\Big(\log\frac{1}{|y|}\Big)^{-\frac{s}{n}(1+\delta)}|x-y|^{s-n}\,dy\\ &\ge c'\int_{2|x|\le|y|\le 1/e}|y|^{-s}\Big(\log\frac{1}{2|x|}\Big)^{-\frac{s}{n}(1+\delta)}|y|^{s-n}\,dy\\ &= c''\Big(\log\frac{1}{2e|x|}\Big)\Big(\log\frac{1}{2|x|}\Big)^{-\frac{s}{n}(1+\delta)}\end{aligned}$$

which tends to infinity as $|x|\to 0$, since δ is so small so that $1-\frac{s}{n}(1+\delta)>0$. Consequently, $\mathcal{I}_s(h)\notin L^\infty$.

Exercises

5.1.1. Let z_1,z_2 be complex numbers satisfying $z_1+\overline{z_2}$ is real. Find a $w\in\mathbf{C}$ such that for all φ in $\mathscr{S}(\mathbf{R}^n)$ we have

$$\int_{\mathbf{R}^n}\mathcal{I}_{z_1}(\varphi)(x)\,\overline{\mathcal{I}_{z_2}(\varphi)(x)}\,dx=\big\|(-\Delta)^w\varphi\big\|_{L^2(\mathbf{R}^n)}^2.$$

5.1.2. Let $1\le q\le\infty$ and $z\in\mathbf{C}$ have positive real part. Prove that for $\varphi\in\mathscr{S}(\mathbf{R}^n)$ we have