Exercises

4.6.1. Prove that the set

$$\widehat{\mathscr{S}_{0,\dots,0}} = \Big\{\varphi \in \mathscr{S}(\mathbf{R}^n):\ \widehat{\varphi} \in \mathscr{C}_0^\infty \text{ and } \min_{1\le j\le n} \operatorname{dist}\big[\operatorname{supp}(\widehat{\varphi}), \{x\in\mathbf{R}^n:\ x_j = 0\}\big] > 0\Big\}$$

is dense in $L^p(\mathbf{R}^n)$ when $1 < p < \infty$. $\big[$*Hint:* Mimic the proof of Proposition 2.5.4.$\big]$

4.6.2. Let $1 < p < \infty$. Prove that there is a constant $C_{n,p}$ such that for any finite subset S of $\mathbf{Z}^n$ and every $f_{\boldsymbol{j}}$ in $L^p(\mathbf{R}^n)$, $\boldsymbol{j} \in \mathbf{Z}^n$, we have

$$\Big\|\sum_{\boldsymbol{j}\in S} \Delta^\sharp_{\boldsymbol{j}}(f_{\boldsymbol{j}})\Big\|_{L^p} \le c_{n,p} \Big\|\Big(\sum_{\boldsymbol{j}\in S}|f_{\boldsymbol{j}}|^2\Big)^{\frac12}\Big\|_{L^p}$$

and

$$\Big\|\sum_{\boldsymbol{j}\in S} \Delta^\sharp_{\boldsymbol{j}}(f_{\boldsymbol{j}})\Big\|_{L^p} \le c_{n,p} \Big\|\Big(\sum_{\boldsymbol{j}\in S}|\Delta^\sharp_{\boldsymbol{j}}(f_{\boldsymbol{j}})|^2\Big)^{\frac12}\Big\|_{L^p}.$$

Conclude that there is a constant $C_{n,p}$ such that for every $f \in L^p(\mathbf{R}^n)$ we have

$$\Big\|\sum_{\boldsymbol{j}\in S} \Delta^\sharp_{\boldsymbol{j}}(f)\Big\|_{L^p} \le C_{n,p}\big\|f\big\|_{L^p}.$$

4.6.3. Suppose that $\{m_{\boldsymbol{j}}\}_{\boldsymbol{j}\in\mathbf{Z}^n}$ is a sequence of bounded functions supported in the sets $R_{\boldsymbol{j}}$ defined in (4.6.5). Let $T_{\boldsymbol{j}}(f) = (\widehat{f}\, m_{\boldsymbol{j}})^\vee$ be the multiplier operator associated with $m_{\boldsymbol{j}}$. Let $1 < p < \infty$. Assume that there is a constant A_p for all sequences of functions $\{f_{\boldsymbol{j}}\}_{\boldsymbol{j}\in\mathbf{Z}^n}$ with $f_{\boldsymbol{j}} \in L^p(\mathbf{R}^n)$ the vector-valued inequality

$$\Big\|\Big(\sum_{\boldsymbol{j}\in\mathbf{Z}^n}|T_{\boldsymbol{j}}(f_{\boldsymbol{j}})|^2\Big)^{\frac12}\Big\|_{L^p(\mathbf{R}^n)} \le A_p \Big\|\Big(\sum_{\boldsymbol{j}\in\mathbf{Z}^n}|f_{\boldsymbol{j}}|^2\Big)^{\frac12}\Big\|_{L^p(\mathbf{R}^n)}$$

is valid. Prove there is a $C_{p,n} > 0$ such that for all finite subsets S of $\mathbf{Z}^n$ we have

$$\Big\|\sum_{\boldsymbol{j}\in S} m_{\boldsymbol{j}}\Big\|_{\mathscr{M}_p} \le C_{p,n} A_p.$$

4.6.4. Fix $\theta \in \mathbf{S}^{n-1}$. For $j \in \mathbf{Z}$ define sets $S^\theta_j = \{\xi \in \mathbf{R}^n:\ 2^j \le |\xi\cdot\theta| < 2^{j+1}\}$ and operators $T_{S^\theta_j}(f) = (\widehat{f}\chi_{S^\theta_j})^\vee$ initially on $\mathscr{S}(\mathbf{R}^n)$ and later extended on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$. Prove that for any $g \in L^p(\mathbf{R}^n)$ we have

$$\big\|g\big\|_{L^p(\mathbf{R}^n)} \approx \Big\|\Big(\sum_{j\in\mathbf{Z}}|T_{S^\theta_j}(g)|^2\Big)^{\frac12}\Big\|_{L^p(\mathbf{R}^n)}.$$

$\big[$*Hint:* Consider first the case $\theta = e_1$ and then apply a rotation.$\big]$