$$\sup_{y\in\mathbf{R}^n\setminus\{0\}}\int_{|x|\geq 2|y|}\Big(\sum_{j\in\mathbf{Z}}|K_j(x-y)-K_j(x)|^2\Big)^{\frac{1}{2}}\,dx\leq A_2<\infty,$$

$$\sup_{j\in\mathbf{Z}}\sup_{\varepsilon>0}\sup_{R>\varepsilon}\bigg|\int_{\varepsilon\leq|y|\leq R}K_j(y)\,dy\bigg|\leq A_3<\infty,$$

and there is a sequence $\varepsilon_k\downarrow 0$ and numbers L_j (for each $j\in\mathbf{Z}$) such that

$$\lim_{\varepsilon_k\downarrow 0}\int_{\varepsilon_k\leq|y|\leq 1}K_j(y)\,dy=L_j\,.$$

Suppose that the functions K_j coincide with tempered distributions W_j that satisfy

$$\sum_{j\in\mathbf{Z}}|\widehat{W_j}(\xi)|^2\leq B^2\,,\qquad \xi\in\mathbf{R}^n.$$

Prove that the operator

$$f\to\Big(\sum_{j\in\mathbf{Z}}|K_j*f|^2\Big)^{\frac{1}{2}}$$

maps $L^p(\mathbf{R}^n)$ to itself and is of weak type $(1,1)$ with norms at most $C_{n,p}(A_2+B)$. $\big[$*Hint:* Notice that (4.4.12), (4.4.13), (4.4.14), and (4.4.15) hold by assumption.$\big]$

4.5 Reverse Littlewood–Paley Inequalities

The focus of this section is to study the reverse inequality of that in Theorem 4.4.2. We recall that $\langle f,\varphi\rangle$ denotes the action of a tempered distribution f on a Schwartz function φ, and $\langle f,\varphi\rangle$ coincides with the standard Lebesgue integral $\int_{\mathbf{R}^n}f(x)\varphi(x)\,dx$ if f happens to be an L^p function for some $1\leq p\leq\infty$.

We can extend the definition of the Littlewood–Paley operator Δ_j^{Ω} to tempered distributions whenever Ω is a Schwartz function. Precisely, if $\Omega\in\mathscr{S}(\mathbf{R}^n)$ and f in $\mathscr{S}'(\mathbf{R}^n)$, then $\Delta_j^{\Omega}(f)$ is well defined as the convolution $\Omega_{2^{-j}}*f$. This convolution always produces a smooth function (Theorem 2.7.1).

Recall the reflection $\widetilde{\Omega}$ of a function Ω is given by $\widetilde{\Omega}(x)=\Omega(-x)$ for $x\in\mathbf{R}^n$. We begin by identifying the transpose operator of Δ_j^{Ω} for a function Ω.

Proposition 4.5.1. (a) *If Ω lies in $L^1(\mathbf{R}^n)$ and f in $L^p(\mathbf{R}^n)$, $1\leq p\leq\infty$, then we have*

$$\big\langle f,\Delta_j^{\Omega}(g)\big\rangle=\big\langle\Delta_j^{\widetilde{\Omega}}(f),g\big\rangle\qquad\text{whenever } g\in L^{p'}(\mathbf{R}^n).\tag{4.5.1}$$

(b) *For any $f\in\mathscr{S}'(\mathbf{R}^n)$ and $\Omega\in\mathscr{S}(\mathbf{R}^n)$ we have*

$$\big\langle f,\Delta_j^{\Omega}(\varphi)\big\rangle=\big\langle\Delta_j^{\widetilde{\Omega}}(f),\varphi\big\rangle\qquad\text{whenever } \varphi\in\mathscr{S}(\mathbf{R}^n).\tag{4.5.2}$$

Thus, in these senses, the transpose of the operator Δ_j^{Ω} is $\Delta_j^{\widetilde{\Omega}}$.