4.4.4. Let Ψ,Φ be as (4.4.23) and (4.4.25). Prove the operator identities on L^2

$$\sum_{j\in\mathbf{Z}} \Delta_j^{\Psi} = I \quad \text{and} \quad S_0^{\Phi} + \sum_{j=1}^{\infty} \Delta_j^{\Psi} = I\,.$$

4.4.5. Let Ψ be a Schwartz function whose Fourier transform vanishes in a neighborhood of the origin and let $\varphi \in \mathscr{S}(\mathbf{R}^n)$. Prove that for any $M>0$ there is a constant $C_M = C_{M,n,\Psi,\varphi}$ such that

$$\sum_{j\in\mathbf{Z}} |\Delta_j^{\Psi}(\varphi)(x)| \le \frac{C_M}{(1+|x|)^M}\,.$$

Conclude that if Ψ is an (4.4.23), then for all $0<p\le\infty$ one has

$$\sum_{|j|\le N} \Delta_j^{\Psi}(\varphi) \to \varphi \qquad \text{in } L^p \text{ as } N\to\infty.$$

$\big[$*Hint:* Use the estimates $|(\Psi_{2^{-j}} * \varphi)(x)| \le C_{M,n} 2^{\min(0,j)n}(1+2^{\min(0,j)}|x|)^{-M}$ and $|(\Psi_{2^{-j}} * \varphi)(x)| \le C_{M,L,n} 2^{-Lj}(1+|x|)^{-M}$ for any $L,M \in \mathbf{Z}^+ \cup \{0\}$. These are consequences of Theorems 7.1.1 and 3.3.5 and the second estimate uses that Ψ has vanishing moments of all orders. Last assertion: start with $p=\infty$.$\big]$

4.4.6. Let Ψ be a Schwartz function that satisfies (4.4.23). Let $1<p<\infty$. Prove that for $g \in L^p(\mathbf{R}^n)$ we have

$$\lim_{N\to\infty} \Big\| \sum_{|j|\le N} \Delta_j^{\Psi}(g) - g \Big\|_{L^p} = 0.$$

$\big[$*Hint:* Use Exercise 4.4.5.$\big]$

4.4.7. Let m be a bounded function on $\mathbf{R}^n$ that is supported in the annulus $1 \le |\xi| \le 2$ and define $T_j(f) = \big(\widehat{f}\, m(2^{-j}(\cdot))\big)^{\vee}$. Suppose that the square function

$$f \mapsto \Big(\sum_{j\in\mathbf{Z}} |T_j(f)|^2 \Big)^{1/2}$$

is bounded on $L^p(\mathbf{R}^n)$ for some $1<p<\infty$. Show that there is a constant $C_{p,n}$ such that for every finite subset S of the integers and every $f \in L^p(\mathbf{R}^n)$ we have

$$\Big\| \sum_{j\in S} T_j(f) \Big\|_{L^p(\mathbf{R}^n)} \le C_{p,n} \|f\|_{L^p(\mathbf{R}^n)}.$$

4.4.8. Prove the following generalization of Theorem 4.4.2. Let $A_1, A_2 > 0$. Suppose that K_j, $j \in \mathbf{Z}$ are locally integrable functions on $\mathbf{R}^n \setminus \{0\}$ that satisfy

$$\Big(\sum_{j\in\mathbf{Z}} |K_j(x)|^2 \Big)^{\frac12} \le \frac{A_1}{|x|^n}\,, \qquad x \ne 0,$$