(b) Here $\vec{G} = \{g_j\}_{j=1}^N$ is a row vector but $\vec{S}(\vec{G})$ is now a row vector consisting of columns of length L. The jth column of $\vec{S}(\vec{G})$ is $(K_i * g_j)_{i=1}^L$. In this case we obtain (4.3.9) by repeating the proof of estimate (4.1.8) and replacing any appearance of ℓ_M^s in the range by $\ell_N^q(\ell_L^\infty)$. Note that the domain $\ell_N^q(\ell_1^1) = \ell_N^q$ remains unchanged.

The transpose operator $\vec{T}^t$ of $\vec{T}$ has kernel $(\widetilde{K_1}, \dots, \widetilde{K_L})^t$ and the transpose operator $\vec{S}^t$ of $\vec{S}$ has kernel $(\widetilde{K_1}, \dots, \widetilde{K_L})$, and these kernels obviously satisfy (4.3.5) and (4.3.6). So, modulo the reflection of the K_j, the operators $\vec{T}$ and $\vec{S}$ are transposes of one another. Next we interpolate between $\vec{T} : L^r(\mathbf{R}^n, \ell_N^q(\ell_L^1)) \to L^r(\mathbf{R}^n, \ell_N^q)$ and estimate (4.3.7). Using Exercise 1.3.7, we obtain for $1 < p < r$

$$\big\|\vec{T}\big\|_{L^p(\mathbf{R}^n,\ell_N^q(\ell_L^1)) \to L^p(\mathbf{R}^n,\ell_N^q)} \le 2C_n' \Big(\frac{p}{p-1} + \frac{p}{r-p}\Big)^{\frac{1}{p}} (A_2 + B_\star). \tag{4.3.11}$$

If $r = \infty$, the constant in (4.3.11) raised to the power $1/p$ is bounded by $C(p) = p(p-1)^{-1}$. Now if $r < \infty$, notice that $\vec{T}^t$ maps $L^{r'}(\mathbf{R}^n, \ell_N^{q'})$ to $L^{r'}(\mathbf{R}^n, \ell_N^{q'}(\ell_L^\infty))$ with bound $B_\star$. As the kernel of $\vec{T}^t$ satisfies the same estimates as that of $\vec{S}$, it follows that $\vec{T}^t$ also admits a bounded extension from $L^1(\mathbf{R}^n, \ell_N^{q'})$ to $L^{1,\infty}(\mathbf{R}^n, \ell_N^{q'}(\ell_L^\infty))$ with bound at most $C_n'(A_2 + B_\star)$. By interpolation (Exercise 1.3.7) we obtain for $1 < p' < r'$

$$\big\|\vec{T}^t\big\|_{L^{p'}(\mathbf{R}^n,\ell_N^{q'}) \to L^{p'}(\mathbf{R}^n,\ell_N^{q'}(\ell_L^\infty))} \le 2C_n' \Big(\frac{p'}{p'-1} + \frac{p'}{r'-p'}\Big)^{\frac{1}{p'}} (A_2 + B_\star). \tag{4.3.12}$$

Estimates (4.3.11) and (4.3.12) imply statements analogous to (4.1.19) and (4.1.20) with ℓ_N^q replaced by $\ell_N^q(\ell_L^1)$ and ℓ_M^s replaced by ℓ_N^q. The rest of the argument proceeds as that in the proof of Theorem 4.1.1 A completely analogous argument is also valid for $\vec{S}$. Combining these ingredients completes the proof. □

We now pass to an application which extends the discussion in Example 4.2.4. Let $\Phi(x) = (1+|x|)^{-n-\gamma}$ be as in that example. For a fixed odd positive integer L let $\{t_1, \dots, t_L\} = \{2^{-[L/2]}, 2^{-[L/2]+1}, \dots, 2^{[L/2]}\}$. We consider the $L \times 1$ matrix $\vec{K}(x) = (\Phi_{t_1}(x), \dots, \Phi_{t_L}(x))^t$ defined on $\mathbf{R}^n$. This matrix can be viewed as the operator

$$(a_1, a_2, \dots, a_N) \mapsto \begin{pmatrix} \Phi_{t_1}(x) \\ \Phi_{t_2}(x) \\ \vdots \\ \Phi_{t_L}(x) \end{pmatrix} \big(a_1 \ a_2 \cdots \ a_N\big) = \begin{pmatrix} \Phi_{t_1}(x)a_1 & \Phi_{t_1}(x)a_2 & \dots & \Phi_{t_1}(x)a_N \\ \Phi_{t_2}(x)a_1 & \Phi_{t_2}(x)a_2 & \dots & \Phi_{t_2}(x)a_N \\ \vdots & \vdots & \vdots & \vdots \\ \Phi_{t_L}(x)a_1 & \Phi_{t_L}(x)a_2 & \dots & \Phi_{t_L}(x)a_N \end{pmatrix},$$

which maps ℓ_N^r to $\ell_N^r(\ell_L^\infty)$ with norm

$$\big\|\vec{K}(x)\big\|_{\ell_N^r \to \ell_N^r(\ell_L^\infty)} = \sup_{1 \le i \le L} |\Phi_{t_i}(x)|.$$

Properties (4.3.5) and (4.3.6) are proved via the arguments yielding (4.2.7) and (4.2.8), respectively. Additionally, for $1 < r < \infty$, the estimate