1.2.3. (Fatou's lemma for weak L^p spaces) Let $f_k \geq 0$ be measurable functions on a measure space (X,μ) and $0 < p < \infty$. Prove that

$$\big\| \liminf_{k\to\infty} f_k \big\|_{L^{p,\infty}} \leq \liminf_{k\to\infty} \big\| f_k \big\|_{L^{p,\infty}} .$$

$\big[$*Hint:* Set $g_k = \inf\{f_l :\ l \geq k\}$ and use the previous exercise.$\big]$

1.2.4. Suppose f and f_k are measurable functions on $\mathbf{R}^n$. Prove that if $|f| \leq \liminf_{k\to\infty} |f_k|$ a.e., then $D_f \leq \liminf_{k\to\infty} D_{f_k}$.

1.2.5. Let $0 < p_0 < p < p_1 \leq \infty$ and let $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ for some $\theta \in (0,1)$. Prove

$$\big\| f \big\|_{L^{p,\infty}} \leq \big\| f \big\|_{L^{p_0,\infty}}^{1-\theta} \big\| f \big\|_{L^{p_1,\infty}}^{\theta} .$$

1.2.6. Let (X,μ) be a measure space and let E be a subset of X with $\mu(E) < \infty$. Assume that f is in $L^{p,\infty}(X,\mu)$ for some $0 < p < \infty$.
(a) Show that for $0 < q < p$ we have

$$\int_E |f(x)|^q \, d\mu(x) \leq \frac{p}{p-q} \mu(E)^{1-\frac{q}{p}} \big\| f \big\|_{L^{p,\infty}}^q .$$

(b) Prove that if $\mu(X) < \infty$ and $0 < q < p < \infty$, then

$$L^p(X,\mu) \subsetneqq L^{p,\infty}(X,\mu) \subsetneqq L^q(X,\mu).$$

(c) Conclude that $L^{p,\infty}(\mathbf{R}^n)$ is contained in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$ when $p > 1$.

1.2.7. (Hölder's inequality for weak L^p spaces) Let f_1 be in $L^{p_1,\infty}$ and f_2 be in $L^{p_2,\infty}$ of a measure space (X,μ) where $0 < p_1, p_2 < \infty$. Given $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, prove that

$$\big\| f_1 f_2 \big\|_{L^{p,\infty}} \leq \big[(p_2/p_1)^{\frac{p_1}{p_1+p_2}} + (p_1/p_2)^{\frac{p_2}{p_1+p_2}} \big]^{\frac{1}{p}} \big\| f_1 \big\|_{L^{p_1,\infty}} \big\| f_2 \big\|_{L^{p_2,\infty}} .$$

Observe that the preceding inequality also extends to the case where p_1, p_2 equal ∞. $\big[$*Hint:* For $\|f_j\|_{L^{p_j,\infty}} = 1$, $j = 1,2$, use $D_{f_1 f_2}(\lambda) \leq \mu(\{|f_1| > \lambda/s\}) + \mu(\{|f_2| > s\}) \leq (s/\lambda)^{p_1} + (1/s)^{p_2}$ and minimize over $s > 0$.$\big]$

1.2.8. Let $f \in L^1([0,\infty))$ and $g \in L^1((-\infty,0])$. Prove that the function

$$x \mapsto \int_{\mathbf{R}} f(x+t) g(x-t) \frac{dt}{t}$$

lies in $L^{1/2,\infty}(\mathbf{R})$ with quasi-norm bounded by $4\|f\|_{L^1}\|g\|_{L^1}$. $\big[$*Hint:* Control this function pointwise by $|x|^{-1}G(x)$, for some $G \geq 0$ with $\|G\|_{L^1} \leq \frac{1}{2}\|f\|_{L^1}\|g\|_{L^1}$.$\big]$