3.7.5. For $1 < p < \infty$ let $A_p = \|H\|_{L^p \to L^p}$ and $B_p = \|\mathcal{M}\|_{L^p \to L^p}$, where H is the Hilbert transform and $\mathcal{M}$ is the (centered) Hardy–Littlewood maximal function on $\mathbf{R}$. Let Q be the conjugate Poisson kernel. Prove that for any $f \in L^p(\mathbf{R})$ we have

$$\sup_{\varepsilon>0} \big\|Q_\varepsilon * f\big\|_{L^p(\mathbf{R})} \le A_p \big\|f\big\|_{L^p(\mathbf{R})},$$
$$\big\|\sup_{\varepsilon>0} |Q_\varepsilon * f|\big\|_{L^p(\mathbf{R})} \le A_p B_p \big\|f\big\|_{L^p(\mathbf{R})}.$$

[*Hint:* Use Lemma 3.7.2 and Proposition 2.5.3.]

3.7.6. Let $2 \le p < \infty$. Prove that for any real-valued function $f \in L^p(\mathbf{R})$ we have

$$\sup_{\varepsilon>0} \big\|(P_\varepsilon * f) + i\,(Q_\varepsilon * f)\big\|_{L^p(\mathbf{R})} \le (1 + A_p^2)^{\frac{1}{2}}\, \big\|f\big\|_{L^p(\mathbf{R})},$$

where $A_p = \|H\|_{L^p \to L^p}$ and H is the Hilbert transform.
[*Hint:* Use the preceding exercise and the subadditivity of the $L^{p/2}$ norm.]

3.7.7. On $\mathbf{R}^n$ define the jth *conjugate Poisson kernel* $Q^{(j)}$ by

$$Q^{(j)}(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{x_j}{(|x|^2+1)^{\frac{n+1}{2}}}, \qquad 1 \le j \le n.$$

Let $Q_y^{(j)}$ be the L^1 dilation of $Q^{(j)}$ for $y > 0$. Prove that

$$(Q_y^{(j)})^\wedge(\xi) = -i \frac{\xi_j}{|\xi|} e^{-2\pi y |\xi|}.$$

Conclude that $R_j(P_y) = Q_y^{(j)}$ and that for all f in $L^p(\mathbf{R}^n)$, $1 < p < \infty$, we have $R_j(f) * P_y = f * Q_y^{(j)}$ for $y > 0$.

3.7.8. Let $f \in L^p(\mathbf{R}^n)$ where $1 < p < \infty$. Prove that the truncated Riesz transforms $R_j^{(\varepsilon)}(f)$ converge to $R_j(f)$ in L^p and a.e. as $\varepsilon \to 0$.
[*Hint:* Using Exercise 3.7.7, write $R_j^{(\varepsilon)}(f) = R_j^{(\varepsilon)}(f) - f * Q_\varepsilon^{(j)} + R_j(f) * P_\varepsilon$ and then apply the idea in Theorem 3.7.4.]

3.7.9. Let η be an even smooth function on the real line such that $\eta(t) = 1$ for $|t| \ge 1$ and η vanishes for $|t| \le \frac{1}{2}$. Define the *smoothly truncated Hilbert transform* (associated with η) acting on a function $f \in L^p(\mathbf{R})$ $(1 < p < \infty)$ by

$$H_\eta^{(\varepsilon)}(f)(x) = \int_{\mathbf{R}} f(x-t) \frac{\eta(t/\varepsilon)}{t}\, dt.$$

Given $1 < p < \infty$ and $f \in L^p(\mathbf{R})$, prove that $H_\eta^{(\varepsilon)}(f) \to H(f)$ in L^p and a.e. as $\varepsilon \to 0$.