Proof. Let $f \in \mathscr{S}(\mathbf{R}^n)$. We express a general singular integral T_Ω with Ω odd and bounded on the sphere as follows. For given $x \in \mathbf{R}^n$ and $\varepsilon > 0$ we write

$$\begin{aligned}\int_{|y|\geq\varepsilon} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\,dy &= +\int_{\mathbf{S}^{n-1}} \Omega(\theta) \int_{r=\varepsilon}^{\infty} f(x-r\theta)\,\frac{dr}{r}\,d\theta \\ &= \int_{\mathbf{S}^{n-1}} \Omega(-\theta) \int_{r=\varepsilon}^{\infty} f(x+r\theta)\,\frac{dr}{r}\,d\theta, \\ &= -\int_{\mathbf{S}^{n-1}} \Omega(\theta) \int_{r=\varepsilon}^{\infty} f(x+r\theta)\,\frac{dr}{r}\,d\theta,\end{aligned}$$

where the first identity follows by switching to polar coordinates, the second one uses the change of variables $\theta \mapsto -\theta$, and the third one expresses that Ω is an odd function on the sphere. Averaging the first and third identities on the right, we obtain

$$\begin{aligned}&\int_{|y|\geq\varepsilon} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\,dy \\ &\qquad= \frac{\pi}{2}\int_{\mathbf{S}^{n-1}} \Omega(\theta)\,\frac{1}{\pi}\int_{r=\varepsilon}^{\infty} \frac{f(x-r\theta)-f(x+r\theta)}{r}\,dr\,d\theta. \qquad (3.6.10)\end{aligned}$$

We write $\frac{1}{r}\big(f(x-r\theta)-f(x+r\theta)\big) = -\int_{-1}^{1} \nabla f(x-sr\theta)\cdot\theta\,ds$ and we note that the first expression has rapid decay in r (for sufficiently large r) and that the second expression is bounded. Then by the Lebesgue dominated convergence theorem, we can pass the limit as $\varepsilon \to 0^+$ inside the integral over $\mathbf{S}^{n-1}$ in (3.6.10). This yields

$$\begin{aligned}T_\Omega(f)(x) &= \frac{\pi}{2}\int_{\mathbf{S}^{n-1}} \Omega(\theta) \lim_{\varepsilon\to 0^+} \frac{1}{\pi}\int_{r=\varepsilon}^{\infty} \frac{f(x-r\theta)-f(x+r\theta)}{r}\,dr\,d\theta \\ &= \frac{\pi}{2}\int_{\mathbf{S}^{n-1}} \Omega(\theta) \lim_{\varepsilon\to 0^+} \frac{1}{\pi}\int_{|r|\geq\varepsilon} f(x-r\theta)\frac{dr}{r}\,d\theta \\ &= \frac{\pi}{2}\int_{\mathbf{S}^{n-1}} \Omega(\theta)\,\mathscr{H}_\theta(f)(x)\,d\theta \qquad (3.6.11)\end{aligned}$$

for $x \in \mathbf{R}^n$ and $f \in \mathscr{S}(\mathbf{R}^n)$. The boundedness of T_Ω on $L^p(\mathbf{R}^n)$ to itself is then a straightforward consequence of (3.6.11) via Minkowski's integral inequality. □

Exercises

3.6.1. Assume that T is a linear operator acting on measurable functions on $\mathbf{R}^n$ such that whenever a function f is supported in a cube Q, then $T(f)$ is supported in $Q^* = \rho Q$ for some $\rho > 1$. Suppose that T maps L^2 to L^2 with norm B. Prove that T extends to a bounded operator from L^1 to $L^{1,\infty}$ with norm a constant multiple of B.

3.6.2. Let K satisfy (3.3.3) and (3.3.4) for some $A_1, A_2 > 0$. Let $W \in \mathscr{S}'(\mathbf{R}^n)$ be associated as in (3.3.8) with a sequence $\delta_j \in (0,1)$ that tends to zero and let T be the operator given by convolution with W. Suppose that T maps $L^\infty(\mathbf{R}^n)$ to itself