Corollary 3.6.2. *Let K be a function on $\mathbf{R}^n \setminus \{0\}$ that satisfies* (3.3.3), (3.3.4)*, and* (3.3.5) *for some $A_1, A_2, A_3 < \infty$. Let W be the distribution associated with K as in* (3.3.8)*. Then T has an extension on L^p for all $p \in [1, \infty)$ that satisfies* (3.6.1) *and* (3.6.2) *with $B = 9\omega_{n-1}A_1 + A_2 + A_3$.*

Proof. Conditions (3.3.3), (3.3.4), and (3.3.5) imply that T is L^2 bounded with bound $B \le 9\omega_{n-1}A_1 + A_2 + A_3$ in view of Theorem 3.4.2. Then the L^2 hypothesis in Theorem 3.6.1 also holds and the conclusion follows. □

Corollary 3.6.3. *The Hilbert transform and the Riesz transforms are bounded from L^1 to $L^{1,\infty}$ and from L^p for all $1 < p < \infty$ with bounds $C(n)\max((p-1)^{-1}, p)$.*

Proof. This is a direct consequence of Corollary 3.6.2. □

Having established the L^p boundedness of the Hilbert transform and of the Riesz transforms for $1 < p < \infty$, we turn to general odd singular integrals of homogeneous type. We begin with the observation that the Hilbert transform acting on the first variable on $\mathbf{R}^n$ composed with the identity operator in the remaining variables

$$\mathscr{H}_{e_1}(f)(x_1, x_2, \dots, x_n) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{|t| \ge \varepsilon} f(x_1 - t, x_2, \dots, x_n) \frac{dt}{t}, \tag{3.6.7}$$

defined for $f \in \mathscr{S}(\mathbf{R}^n)$, is bounded on $L^p(\mathbf{R}^n)$ with bound $C_p = \|H\|_{L^p \to L^p}$. To verify this, we raise the absolute values of both sides in (3.6.7) to the power p and then integrate in x_1. Using the boundedness of the Hilbert transform we estimate the right-hand side by

$$C_p^p \int_{\mathbf{R}} |f(x_1, x_2, \dots, x_n)|^p \, dx_1 .$$

Integrating over the remaining variables implies the conclusion. Next, for a unit vector $\theta \in \mathbf{S}^{n-1}$ we define

$$\mathscr{H}_\theta(f)(x) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{|t| \ge \varepsilon} f(x - t\theta) \frac{dt}{t}, \qquad f \in \mathscr{S}(\mathbf{R}^n), \tag{3.6.8}$$

called the *directional Hilbert transform* in the direction θ. We observe that the following identity is valid for all matrices $A \in O(n)$:

$$\mathscr{H}_{Ae_1}(f)(x) = \mathscr{H}_{e_1}(f \circ A)(A^{-1}x) . \tag{3.6.9}$$

Now given $\theta \in \mathbf{S}^{n-1}$ pick $A \in O(n)$ such that $Ae_1 = \theta$. This implies that

$$\big\|\mathscr{H}_\theta(f)\big\|_{L^p} = \big\|\mathscr{H}_{e_1}(f \circ A) \circ A^{-1}\big\|_{L^p} = \big\|\mathscr{H}_{e_1}(f \circ A)\,\big\|_{L^p} \le C_p \big\|f \circ A\big\|_{L^p} = C_p \big\|f\big\|_{L^p},$$

and this yields that $\mathscr{H}_\theta$ maps $L^p(\mathbf{R}^n)$ to ~~itself~~ $L^p(\mathbf{R}^n)$ uniformly in θ. We use this to obtain boundedness for T_Ω (Definition 3.2.2) when Ω is an odd bounded function. Such functions clearly have vanishing integral.

Corollary 3.6.4. *Let $\Omega \in L^\infty(\mathbf{S}^{n-1})$ be odd and let T_Ω be as in Definition 3.2.2. Then for $1 < p < \infty$, T_Ω admits a bounded extension from $L^p(\mathbf{R}^n)$ to itself with norm at most $\frac{\pi\omega_{n-1}}{2}\|\Omega\|_{L^\infty}\|H\|_{L^p \to L^p}$.*