Thus we proved that

$$\int_{(\bigcup_i Q_i^*)^c} \sum_j |T(b_j)(x)|\,dx \le 2^{n+1}A_2\big\|f\big\|_{L^1},$$

an inequality we use below. Appealing to (3.6.3), we write

$$\begin{aligned}
&\big|\{x\in\mathbf{R}^n : |T(f)(x)|>\alpha\}\big| \\
&\le \Big|\Big\{x\in\mathbf{R}^n:\ |T(g)(x)|>\frac{\alpha}{2}\Big\}\Big| + \Big|\Big\{x\in\mathbf{R}^n:\ |T(b)(x)|>\frac{\alpha}{2}\Big\}\Big| \\
&\le \frac{4}{\alpha^2}\big\|T(g)\big\|_{L^2}^2 + \Big|\bigcup_i Q_i^*\Big| + \Big|\Big\{x\notin\bigcup_i Q_i^*:\ |T\big(\sum_j b_j\big)(x)|>\frac{\alpha}{2}\Big\}\Big| \\
&\le \frac{4}{\alpha^2}\big\|T(g)\big\|_{L^2}^2 + \Big|\bigcup_i Q_i^*\Big| + \Big|\Big\{x\notin\bigcup_i Q_i^*:\ \sum_j |T(b_j)(x)|>\frac{\alpha}{2}\Big\}\Big| \\
&\le \frac{4}{\alpha^2}B^2\big\|g\big\|_{L^2}^2 + \sum_i |Q_i^*| + \frac{2}{\alpha}\int_{(\bigcup_i Q_i^*)^c}\sum_j |T(b_j)(x)|\,dx \\
&\le \frac{4}{\alpha^2}2^nB^2(\gamma\alpha)\big\|f\big\|_{L^1} + (2\sqrt{n})^n\frac{\big\|f\big\|_{L^1}}{\gamma\alpha} + \frac{2}{\alpha}2^{n+1}A_2\big\|f\big\|_{L^1} \\
&\le \left(\frac{(2^{n+1}B\gamma)^2}{2^n\gamma} + \frac{(2\sqrt{n})^n}{\gamma} + 2^{n+2}A_2\right)\frac{\big\|f\big\|_{L^1}}{\alpha}.
\end{aligned}$$

Choosing $\gamma = B^{-1}$, we deduce estimate (3.6.1) with $C_n' = (2\sqrt{n})^n + 2^{n+2}$.

By the density argument discussed at the beginning of the proof, T is well defined on L^1, and thus on L^p which is contained in $L^1 + L^2$ for $1<p<2$. Using Theorem 1.3.3 (Marcinkiewicz's interpolation theorem) we obtain that

$$\big\|T\big\|_{L^p\to L^p} \le 2\left(\frac{p}{p-1}+\frac{p}{2-p}\right)C_n'(A_2+B), \qquad 1<p<2. \tag{3.6.4}$$

We now observe that the transpose operator T^t of T has kernel $K^t(x) = K(-x)$ which also satisfies (3.3.3) and (3.3.4) for some $A_1, A_2 < \infty$ and moreover T^t maps $L^2(\mathbf{R}^n)$ to itself with the same norm B. Then T^t satisfies (3.6.4), which implies

$$\big\|T\big\|_{L^p\to L^p} \le 2\left(\frac{p'}{p'-1}+\frac{p'}{2-p'}\right)C_n'(A_2+B), \qquad 2<p<\infty. \tag{3.6.5}$$

Now we employ Theorem 1.3.3 to interpolate between $L^{5/4}$ and L^5. We obtain

$$\big\|T\big\|_{L^p\to L^p} \le 2\left(\frac{p}{p-\frac{5}{4}}+\frac{p}{5-p}\right)C_n''(A_2+B), \qquad \frac{5}{4}<p<5. \tag{3.6.6}$$

For $1<p\le\frac{3}{2}$, we use (3.6.4) to obtain $\|T\|_{L^p\to L^p} \le \frac{4p}{p-1}C_n'(A_2+B)$. For $3\le p<\infty$, (3.6.5) yields the bound $\|T\|_{L^p\to L^p} \le 4\,p\,C_n'(A_2+B)$. Finally, for $\frac{3}{2}<p<3$ we use (3.6.6) to obtain the bound $\|T\|_{L^p\to L^p} \le \frac{8p}{p-1}C_n''(A_2+B)$. Combining these cases, we deduce (3.6.2). □