Exercises

3.5.1. Let $\alpha > 0$ and $f \in L^1(\mathbf{R}^n)$ which satisfies $|f| \le \alpha$ a.e. What are the functions b and g constructed in Theorem 3.5.2 in this case?

3.5.2. For a given $f \in L^1(\mathbf{R}^n)$, let $\{Q_j^\alpha\}_j$ be the cubes obtained in the CZ decomposition at height $\alpha > 0$. For given $0 < \alpha < \beta < \infty$ prove that

$$\bigcup_j Q_j^\beta \subseteqq \bigcup_i Q_i^\alpha.$$

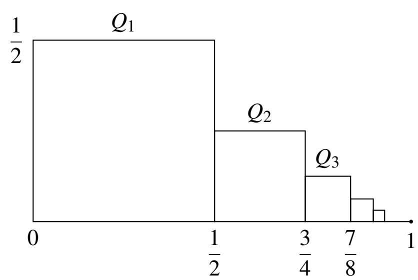
3.5.3. Show that the number of CZ cubes $\{Q_j\}$ in Example 3.5.5 is at most

$$2^{nM} \frac{\|f\|_{L^1}}{\alpha},$$

where $[-2^{N'}, 2^{N'}]^n$ contains the support of f and 2^{-M} is the side length of the smallest dyadic cube appearing in the definition of f.

3.5.4. Prove that finite linear combinations of characteristic functions of dyadic cubes are dense in $L^p(\mathbf{R}^n)$ for any $0 < p < \infty$.

3.5.5. On $\mathbf{R}^2$, let $Q_1 = [0, \frac{1}{2})^2$, and $Q_j = [\frac{1}{2} + \cdots + \frac{1}{2^{j-1}}, \frac{1}{2} + \cdots + \frac{1}{2^j}) \times [0, \frac{1}{2^j})$ for $j \ge 2$. Prove that $(1,0)$ lies in $\overline{\bigcup_{j=1}^\infty Q_j}$ but not in $\bigcup_{j=1}^\infty \overline{Q_j^*}$. Here Q_j^* is a cube with the same center as Q_j but $\ell(Q_j^*) = 2\ell(Q_j)$. Construct a similar example when $\ell(Q_j^*) = 2^N \ell(Q_j)$ for a given $N \in \mathbf{Z}^+$.



3.5.6. (**Calderón–Zygmund decomposition on** L^q) Fix a function $f \in L^q(\mathbf{R}^n)$ for some $1 \le q < \infty$ and let $\alpha > 0$. Then there exist functions g and b on $\mathbf{R}^n$ such that

(1) $f = g + b$.

(2) $\|g\|_{L^q} \le \|f\|_{L^q}$ and $\|g\|_{L^\infty} \le 2^{\frac{n}{q}} \alpha$.

(3) $b = \sum_j b_j$, where each b_j is supported in a cube Q_j. Furthermore, the cubes Q_k and Q_j have disjoint interiors when $j \ne k$.

(4) $\|b_j\|_{L^q}^q \le 2^{n+q} \alpha^q |Q_j|$.