Remark 1.2.7. Note that for $0 < r < \infty$, $\||f|^r\|_{L^p} = \|f\|^r_{L^{pr}}$. One can verify that the same property holds for the weak L^p quasi-norm, i.e., $\||f|^r\|_{L^{p,\infty}} = \|f\|^r_{L^{pr,\infty}}$.

Definition 1.2.8. The space $L^1_{\text{loc}}(\mathbf{R}^n, |\cdot|)$ of *locally integrable* functions is the set of all Lebesgue-measurable functions f on $\mathbf{R}^n$ that satisfy

$$\int_K |f(x)|\,dx < \infty \tag{1.2.7}$$

for any compact subset K of $\mathbf{R}^n$.

Example 1.2.9. (a) $L^p(\mathbf{R}^n)$ is contained in $L^1_{\text{loc}}(\mathbf{R}^n)$ for $1 \le p \le \infty$ (Theorem 1.1.3).
(b) For $0 < p < 1$, $L^p(\mathbf{R}^n) \setminus L^1_{\text{loc}}(\mathbf{R}^n) \neq \emptyset$; i.e., it contains $|x|^{-\frac{n}{p}}\left(\log\frac{1}{|x|}\right)^{-\frac{2}{p}}\chi_{|x|\le\frac{1}{2}}$.
(c) There exist functions in $L^{1,\infty}(\mathbf{R}^n) \setminus L^1_{\text{loc}}(\mathbf{R}^n)$, such as $|x|^{-n}$.
(d) The function $e^{e^{|x|}}$ lies in $L^1_{\text{loc}}(\mathbf{R}^n)$ but not in $L^p(\mathbf{R}^n)$ for any $p > 0$.

Theorem 1.2.10. *For a measurable function f on a σ-finite measure space (X,μ) define*

$$|||\,f\,|||_{L^{p,\infty}} = \sup_{0<\mu(E)<\infty} \mu(E)^{-1+\frac{1}{p}} \int_E |f|\,d\mu\,.$$

Let $1 < p < \infty$. Then $|||\cdot|||_{L^{p,\infty}}$ is a norm on $L^{p,\infty}$ that satisfies

$$\|f\|_{L^{p,\infty}} \le |||\,f\,|||_{L^{p,\infty}} \le \frac{p}{p-1}\|f\|_{L^{p,\infty}}.$$

Proof. Let $E \subsetneqq X$ such that $0 < \mu(E) < \infty$ and let $f \in L^{p,\infty}(X,\mu)$. By Proposition 1.2.3 we write

$$\begin{aligned}\int_E |f|\,d\mu &= \int_0^\infty \mu(\{|f| > \lambda\} \cap E)\,d\lambda \\ &\le \int_0^\infty \min\left(\mu(E), \frac{\|f\|^p_{L^{p,\infty}}}{\lambda^p}\right) d\lambda \\ &= \frac{p}{p-1}\mu(E)^{1-\frac{1}{p}}\|f\|_{L^{p,\infty}}\,,\end{aligned} \tag{1.2.8}$$

which follows by splitting the integral at $\lambda = \|f\|_{L^{p,\infty}}\mu(E)^{-\frac{1}{p}}$. Therefore,

$$|||\,f\,|||_{L^{p,\infty}} \le \frac{p}{p-1}\|f\|_{L^{p,\infty}}.$$

As X is σ-finite, we can write $X = \cup_{N=1}^\infty X_N$ with $X_1 \subsetneqq X_2 \subsetneqq \cdots$ and $\mu(X_N) < \infty$. For $\lambda > 0$ let $E_\lambda = \{|f| > \lambda\}$ and $E^N_\lambda = \{|f| > \lambda\} \cap X_N$. Then we clearly have $\mu(E^N_\lambda) \le \mu(X_N) < \infty$ and $\int_{E^N_\lambda} |f|\,d\mu \ge \lambda\mu(E^N_\lambda)$. Let us fix $\lambda > 0$ and $N \in \mathbf{Z}^+$. If $\mu(E^N_\lambda) > 0$, then

$$|||\,f\,|||_{L^{p,\infty}} = \sup_{0<\mu(E)<\infty} \mu(E)^{-1+\frac{1}{p}} \int_E |f|\,d\mu \ge \mu(E^N_\lambda)^{-1+\frac{1}{p}}\lambda\mu(E^N_\lambda) = \lambda\mu(E^N_\lambda)^{\frac{1}{p}},$$