In general, the difference of two Calderón–Zygmund operators (Definition 3.3.1) associated with the same function K on $\mathbf{R}^n \setminus \{0\}$ and two sequences δ_k^1 and δ_k^2 is cI, where c is a constant satisfying $|c| \le 2A_3$.

Exercises

3.3.1. Prove the equality in (3.3.1).

3.3.2. Let F be a bounded $\mathscr{C}^1$ function on the real line with F' in $L^\infty(\mathbf{R})$ that has the property

$$\sup_{-\infty<A<B<\infty} \left| \int_A^B F(t)\,dt \right| < \infty.$$

Prove that the kernel

$$K(x) = \frac{F(\log|x|)}{|x|^n}, \qquad x \in \mathbf{R}^n \setminus \{0\},$$

satisfies (3.3.3), (3.3.4), and (3.3.5). An example of such a function is $F(t) = \sin t/t$. A multitude of examples arise by taking $F = G'$, where G, G', G'' are bounded.

3.3.3. Let $\delta > 0$ and η be a smooth function on the real line supported in $[-2,2]$ and equal to 1 on the interval $[-1,1]$. Show that the kernels

$$K_1(x) = \frac{\sin(|x|^{-\delta})}{|x|^n}\big(1-\eta(|x|)\big), \qquad K_2(x) = \frac{\sin(|x|^{\delta})}{|x|^n}\eta(|x|),$$

defined on $\mathbf{R}^n \setminus \{0\}$, satisfy (3.3.3), (3.3.2), and (3.3.5).

3.3.4. Suppose that a function K on $\mathbf{R}^n \setminus \{0\}$ satisfies condition (3.3.3) with constant A_1 and condition (3.3.4) with constant A_2. Let $A_1' = A_1 \omega_{n-1} \log 2$.
(a) Show that the functions $K(x)\chi_{|x|\ge\varepsilon}$ also satisfy condition (3.3.4) uniformly in $\varepsilon > 0$ with constant $\max(A_1', A_2)$ in place of A_2.
(b) Use part (a) to obtain that the truncations $K(x)\chi_{|x|<N}$ satisfy (3.3.4) uniformly in $N > 0$, with constant $2\max(A_1', A_2)$.
(c) Deduce from parts (a), (b) that the double truncations $K^{(\varepsilon,N)}(x) = K(x)\chi_{\varepsilon\le|x|<N}$ also satisfy condition (3.3.4) uniformly in $N, \varepsilon > 0$ with constant $2\max(A_1', A_2)$.
$\big[$*Hint:* Part (b): Write $K(x)\chi_{|x|<N} = K(x) - K(x)\chi_{|x|\ge N}$. Part (c): Use $K(x)\chi_{\varepsilon\le|x|<N} = K(x)\chi_{|x|\ge\varepsilon} - K(x)\chi_{|x|\ge N}$.$\big]$

3.3.5. (a) Prove that for all $x, y \in \mathbf{R}^n$ that satisfy $0 \ne x \ne y$ we have

$$\left| \frac{x-y}{|x-y|} - \frac{x}{|x|} \right| \le 2\,\frac{|y|}{|x|}\,.$$

(b) Let Ω be a bounded function with mean value zero on the sphere $\mathbf{S}^{n-1}$. Suppose that for some $\alpha \in (0,1)$, Ω satisfies the *Lipschitz condition*