$(0,1)$ that tends to zero and with a kernel K that satisfies (3.3.3), (3.3.4), and (3.3.5). Precisely it is an operator of the form

$$T^W(\varphi) = \varphi * W, \qquad \varphi \in \mathscr{S}(\mathbf{R}^n).$$

For $\varphi \in \mathscr{S}(\mathbf{R}^n)$ and $x \in \mathbf{R}^n$, we can explicitly write

$$\begin{aligned} T^W(\varphi)(x) &= \lim_{k\to\infty} \int_{|y|\ge\delta_k} K(y)\varphi(x-y)\,dy \qquad (3.3.9)\\ &= L\varphi(x) + \int_{|y|\le 1} K(y)(\varphi(x-y)-\varphi(x))\,dy + \int_{|y|\ge 1} K(y)\varphi(x-y)\,dy. \end{aligned}$$

Example 3.3.2. Let τ be a nonzero real number and let $K(x) = \frac{1}{|x|^{n+i\tau}}$ be defined for $x \neq 0$. Notice that (3.3.3) is clearly satisfied for K and also (3.3.4) is valid, as $|\nabla K(x)| \le |n+i\tau|\,|x|^{-n-1}$. Finally, (3.3.5) is also satisfied, as for $0<\varepsilon<N<\infty$,

$$\left|\int_{\varepsilon<|x|<N} \frac{1}{|x|^{n+i\tau}}\,dx\right| = \omega_{n-1}\left|\frac{N^{-i\tau}-\varepsilon^{-i\tau}}{-i\tau}\right| \le \frac{2\omega_{n-1}}{|\tau|} = A_3\,.$$

Consider the following two sequences $\delta_k^1 = e^{-(2k+1)\pi/\tau}$ and $\delta_k^2 = e^{-2k\pi/\tau}$ indexed by $k=1,2,\dots$ if $\tau>0$ and by $k=-1,-2,-3,\dots$ if $\tau<0$. Both sequences lie in $(0,1)$ and tend to zero. For a Schwartz function φ on $\mathbf{R}^n$ define distributions

$$\langle W^1,\varphi\rangle = \lim_{k\to\infty}\int_{|x|\ge\delta_k^1}\varphi(x)\frac{dx}{|x|^{n+i\tau}} \qquad (3.3.10)$$

and

$$\langle W^2,\varphi\rangle = \lim_{k\to\infty}\int_{|x|\ge\delta_k^2}\varphi(x)\frac{dx}{|x|^{n+i\tau}}. \qquad (3.3.11)$$

We have that

$$\begin{aligned} \int_{\delta_k^1<|x|<1}\frac{1}{|x|^{n+i\tau}}\,dx &= \omega_{n-1}\frac{1^{-i\tau}-(\delta_k^1)^{-i\tau}}{-i\tau} = \omega_{n-1}\frac{1-e^{(2k+1)i\pi}}{-i\tau} = \frac{2i\omega_{n-1}}{\tau}\,,\\ \int_{\delta_k^2<|x|<1}\frac{1}{|x|^{n+i\tau}}\,dx &= \omega_{n-1}\frac{1^{-i\tau}-(\delta_k^2)^{-i\tau}}{-i\tau} = \omega_{n-1}\frac{1-e^{2ki\pi}}{-i\tau} = 0, \end{aligned}$$

and as these expressions are constant for all integers k, they have limits $L_1 = 2i\omega_{n-1}/\tau$ and $L_2 = 0$ as $|k|\to\infty$, respectively. In general, note that the subsequence $\varepsilon_k = e^{-(2k+\alpha)\pi/\tau}$ yields the limit $\omega_{n-1}\frac{1-e^{-i\pi\alpha}}{-i\tau}$ which varies with $\alpha\in[0,1]$.

Notice that both W^1 and W^2 agree on $\mathbf{R}^n\setminus\{0\}$ but are not the same tempered distribution. In fact, it is not hard to see that $W^1-W^2 = c\,\delta_0$, where $c=\frac{2i\omega_{n-1}}{\tau}$. Thus the associated Calderón–Zygmund singular integral operators are T^{W^1} and T^{W^2}, which differ by a constant multiple of the identity operator.