$$|H(\varphi^{(m)})(x)| \le \frac{C}{(1+|x|)^{m+1}}.$$

[*Hint:* Integrate by parts in the second integral in (3.1.6) and use Exercise 1.8.4.]

3.1.6. Let φ be a Schwartz function on the line and let $m \in \mathbf{Z}^+$. Prove that

$$H(\varphi)(x) = -2i \sum_{k=1}^{m} (-1)^k \frac{\widehat{\varphi}^{(k-1)}(0)}{(2\pi i x)^k} + \frac{1}{(2\pi i x)^m} H\Big((2\pi i(\cdot))^m \varphi\Big)(x)$$

for all $x \in \mathbf{R} \setminus \{0\}$. Conclude that

$$\sup_{x\in\mathbf{R}} (1+|x|)^{m+1} |H(\varphi)(x)| < \infty \iff \int_{\mathbf{R}} t^k \varphi(t)\,dt = 0 \text{ for all } k \in \{0,\dots,m-1\}.$$

[*Hint:* Factor $x^m - t^m$ in the identity

$$H(\varphi)(x) - \frac{1}{(2\pi i x)^m} H\Big((2\pi i(\cdot))^m \varphi\Big)(x) = \frac{1}{\pi}\int_{\mathbf{R}} \frac{x^m - t^m}{x^m} \frac{\varphi(t)}{x-t}\,dt.$$

Alternatively, integrate by parts m times in $\int_0^\infty \widehat{\varphi}(\xi) e^{2\pi i x\xi}\,d\xi$ and $\int_{-\infty}^0 \widehat{\varphi}(\xi) e^{2\pi i x\xi}\,d\xi$.]

3.2 Homogeneous Singular Integrals and Riesz Transforms

To extend the definition of the Hilbert transform to higher dimensions, we modify the function $1/|x|^n$ on $\mathbf{R}^n \setminus \{0\}$ in order to have integral zero over the unit sphere. We can draw inspiration from the one-dimensional case if we write

$$\frac{1}{x} = \frac{\operatorname{sgn} x}{|x|} = \frac{\operatorname{sgn}(x/|x|)}{|x|}, \qquad x \in \mathbf{R} \setminus \{0\}, \tag{3.2.1}$$

and we notice that only the values of sgn on $\mathbf{S}^0 = \{-1, 1\}$ play a role. But on $\{-1,1\}$ the function sgn maps -1 to -1 and 1 to 1 and

$$\int_{\mathbf{S}^0} \operatorname{sgn} x\, d\nu = -1 \cdot \nu(\{-1\}) + 1 \cdot \nu(\{1\}) = -1 \cdot 1 + 1 \cdot 1 = 0,$$

where ν here denotes counting measure on $\mathbf{S}^0$. This identity expresses the cancellation of $1/x$ on the line.

To extend this idea to higher dimensions, we suppose that Ω is a bounded function of the unit sphere $\mathbf{S}^{n-1}$ with mean value zero, i.e.,

$$\int_{\mathbf{S}^{n-1}} \Omega(\theta)\, d\theta = 0,$$

where $d\theta$ here denotes surface measure on $\mathbf{S}^{n-1}$. Then we consider the kernel