Corollary 2.1.21. *Let $f \geq 0$ be an integrable function over a cube Q in $\mathbf{R}^n$ and let $\alpha \geq \frac{1}{|Q|}\int_Q f\,dx$. Then there exist disjoint (possibly empty) open subcubes Q_j of Q such that for almost all $x \in Q \setminus \bigcup_j Q_j$ we have $f \leq \alpha$ and (2.1.21) holds for all Q_j.*

Proof. The proof easily follows by a simple modification of Proposition 2.1.20 in which $\mathbf{R}^n$ is replaced by the fixed cube Q. To apply Corollary 2.1.16, we extend f to be zero outside the cube Q. $\square$

See Exercise 2.1.4 for an application of Proposition 2.1.20 involving maximal functions.

Exercises

2.1.1. A positive Borel measure μ on $\mathbf{R}^n$ is called *inner regular* if for any open subset U of $\mathbf{R}^n$ we have $\mu(U) = \sup\{\mu(K):\ K \subsetneqq U,\ K \text{ compact}\}$ and μ is called *locally finite* if $\mu(B) < \infty$ for all balls B.
(a) Let μ be a positive inner regular locally finite measure on $\mathbf{R}^n$ that satisfies the following *doubling condition*: There exists a constant $D(\mu) > 0$ such that for all $x \in \mathbf{R}^n$ and $r > 0$ we have

$$\mu(3B(x,r)) \leq D(\mu)\,\mu(B(x,r)).$$

For $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n,\mu)$ define the uncentered maximal function $M_\mu(f)$ with respect to μ by

$$M_\mu(f)(x) \;=\; \sup_{r>0}\ \sup_{\substack{z:\,|z-x|<r\\ \mu(B(z,r))\neq 0}} \frac{1}{\mu(B(z,r))}\int_{B(z,r)} |f(y)|\,d\mu(y)\,.$$

Show that M_μ maps $L^1(\mathbf{R}^n,\mu)$ to $L^{1,\infty}(\mathbf{R}^n,\mu)$ with constant at most $D(\mu)$ and $L^p(\mathbf{R}^n,\mu)$ to itself with constant at most $2\big(\frac{p}{p-1}\big)^{\frac{1}{p}} D(\mu)^{\frac{1}{p}}$.

(b) Obtain as a consequence a differentiation theorem analogous to Corollary 2.1.16.
$\big[$*Hint:* Part (a): For $f \in L^1(\mathbf{R}^n,\mu)$ show that the set $E_\alpha = \{M_\mu(f) > \alpha\}$ is open. Then use the argument of the proof of Theorem 2.1.6 and the inner regularity of μ.$\big]$

2.1.2. On $\mathbf{R}$ consider the maximal function M_μ of Exercise 2.1.1.
(a) (*W. H. Young*) Prove the following covering lemma. Given a finite set $\mathscr{F}$ of open intervals in $\mathbf{R}$, prove that there exist two subfamilies each consisting of pairwise disjoint intervals such that the union of the intervals in the original family is equal to the union of the intervals of both subfamilies. Use this result to show that the maximal function M_μ of Exercise 2.1.1 maps $L^1(\mu) \to L^{1,\infty}(\mu)$ with constant at most 2.