*Let* $a = \int_{\mathbf{R}^n} K(x)\,dx$. *Then for all* $f \in L^p(\mathbf{R}^n)$ *and* $1 \le p < \infty$, $(f * K_\varepsilon)(x) \to a f(x)$ *for almost all* $x \in \mathbf{R}^n$ *as* $\varepsilon \to 0$.

*Proof.* Use Theorem 1.2.21 instead of Theorem 1.2.19 in the proof of Corollary 2.1.17. □

The following application of the Lebesgue differentiation theorem uses a simple *stopping-time argument*. This is the sort of argument in which a selection procedure stops when it is exhausted at a certain scale and is then repeated at the next scale. A certain refinement of the following proposition is of fundamental importance in the study of singular integrals given in Chapter 5.

**Proposition 2.1.20.** *Given a nonnegative integrable function* $f$ *on* $\mathbf{R}^n$ *and* $\alpha > 0$, *there exists a collection of disjoint (possibly empty) open cubes* $Q_j$ *such that for almost all* $x \in \big(\bigcup_j Q_j\big)^c$ *we have* $f(x) \le \alpha$ *and*

$$\alpha < \frac{1}{|Q_j|}\int_{Q_j} f(t)\,dt \le 2^n \alpha\,. \tag{2.1.21}$$

*Proof.* The proof provides an excellent paradigm of a stopping-time argument. Start by decomposing $\mathbf{R}^n$ as a union of cubes of equal size, whose interiors are disjoint, and whose diameter is so large that $|Q|^{-1}\int_Q f(x)\,dx \le \alpha$ for every $Q$ in this mesh. This is possible since $f$ is integrable and $|Q|^{-1}\int_Q f(x)\,dx \to 0$ as $|Q| \to \infty$. Call the union of these cubes $\mathscr{E}_0$.

Divide each cube in the mesh into $2^n$ congruent cubes by bisecting each of the sides. Call the new collection of cubes $\mathscr{E}_1$. Select a cube $Q$ in $\mathscr{E}_1$ if

$$\frac{1}{|Q|}\int_Q f(x)\,dx > \alpha \tag{2.1.22}$$

and call the set of all selected cubes $\mathscr{S}_1$. Now subdivide each cube in $\mathscr{E}_1 \setminus \mathscr{S}_1$ into $2^n$ congruent cubes by bisecting each of the sides as before. Call this new collection of cubes $\mathscr{E}_2$. Repeat the same procedure and select a family of cubes $\mathscr{S}_2$ that satisfy (2.1.22). Continue this way ad infinitum and call the cubes in $\bigcup_{m=1}^\infty \mathscr{S}_m$ "selected." If $Q$ was selected, then there exists $Q_1$ in $\mathscr{E}_{m-1}$ containing $Q$ that was not selected at the $(m-1)th$ step for some $m \ge 1$. Therefore,

$$\alpha < \frac{1}{|Q|}\int_Q f(x)\,dx \le 2^n \frac{1}{|Q_1|}\int_{Q_1} f(x)\,dx \le 2^n \alpha\,.$$

Now call $F$ the closure of the complement of the union of all selected cubes. If $x \in F$, then there exists a sequence of cubes containing $x$ whose diameter shrinks down to zero such that the average of $f$ over these cubes is less than or equal to $\alpha$. By Corollary 2.1.16, it follows that $f(x) \le \alpha$ almost everywhere in $F$. This proves the proposition. □

In the proof of Proposition 2.1.20 it was not crucial to assume that $f$ was defined on all $\mathbf{R}^n$, but only on a cube. We now give a local version of this result.