If f is locally integrable, then by considering the average of f over the ball $B(x,|x|+R)$, which contains the ball $B(0,R)$, we obtain

$$\mathcal{M}(f)(x) \geq \frac{\int_{B(0,R)} |f(y)|\,dy}{v_n(|x|+R)^n}\,, \tag{2.1.1}$$

for all $x \in \mathbf{R}^n$, where v_n is the volume of the unit ball in $\mathbf{R}^n$. An interesting consequence of (2.1.1) is the following: suppose that $f \neq 0$ on a set of positive measure E, then $\mathcal{M}(f)$ is not in $L^1(\mathbf{R}^n)$. In other words, if f is in $L^1_{\text{loc}}(\mathbf{R}^n)$ and $\mathcal{M}(f)$ is in $L^1(\mathbf{R}^n)$, then $f = 0$ a.e. To see this, integrate (2.1.1) over the ball $\mathbf{R}^n$ to deduce that $\|f\chi_{B(0,R)}\|_{L^1} = 0$ and thus $f(x) = 0$ for almost all x in the ball $B(0,R)$. Since this is valid for all $R = 1,2,3,\dots$, it follows that $f = 0$ a.e. in $\mathbf{R}^n$.

Another remarkable locality property of $\mathcal{M}$ is that if $\mathcal{M}(f)(x_0) = 0$ for some x_0 in $\mathbf{R}^n$, then $f = 0$ a.e. To see this, we take $x = x_0$ in (2.1.1) to deduce that $\|f\chi_{B(0,R)}\|_{L^1} = 0$ and as before we have that $f = 0$ a.e. on every ball centered at the origin, i.e., $f = 0$ a.e. in $\mathbf{R}^n$.

A related analogue of $\mathcal{M}(f)$ is its uncentered version $M(f)$, defined as the supremum of all averages of f over all open balls containing a given point.

Definition 2.1.3. The *uncentered Hardy–Littlewood maximal function* of f,

$$M(f)(x) = \sup_{\substack{\delta>0 \\ |y-x|<\delta}} \operatorname*{Avg}_{B(y,\delta)} |f|\,,$$

is defined as the supremum of the averages of $|f|$ over all open balls $B(y,\delta)$ that contain the point x.

Clearly $\mathcal{M}(f) \leq M(f)$; in other words, M is a larger operator than $\mathcal{M}$. However, $M(f) \leq 2^n \mathcal{M}(f)$ and the boundedness properties of M are identical to those of $\mathcal{M}$.

Example 2.1.4. On $\mathbf{R}$, let f be the characteristic function of the interval $I = [a,b]$. For $x \in (a,b)$, clearly $M(f)(x) = 1$. For $x > b$, a calculation shows that the largest average of f over all intervals $(y-\delta, y+\delta)$ that contain x is obtained when $\delta = \frac{1}{2}(x-a)$ and $y = \frac{1}{2}(x+a)$. Similarly, when $x < a$, the largest average is obtained when $\delta = \frac{1}{2}(b-x)$ and $y = \frac{1}{2}(b+x)$. We conclude that

$$M(f)(x) = \begin{cases} (b-a)/|x-b| & \text{when } x \leq a\,, \\ 1 & \text{when } x \in (a,b)\,, \\ (b-a)/|x-a| & \text{when } x \geq b\,. \end{cases}$$

Observe that M does not have a jump at $x = a$ and $x = b$ and is in fact equal to the function $\big(1 + \frac{\operatorname{dist}(x,I)}{|I|}\big)^{-1}$.

We are now ready to obtain some basic properties of maximal functions. We need the following simple covering lemma.