Chapter 2
Maximal Functions, Fourier Transform, and Distributions

We have already seen that the convolution of a function with a fixed density is a smoothing operation that produces a certain average of the function. Averaging is an important operation in analysis and naturally arises in many situations. The study of averages of functions is better understood by the introduction of the maximal function which is defined as the largest average of a function over all balls containing a fixed point. Maximal functions are used to obtain almost everywhere convergence for certain integral averages and play an important role in this area, which is called differentiation theory. Although maximal functions do not preserve qualitative information about the given functions, they maintain crucial quantitative information, a fact of great importance in the subject of Fourier analysis.

Another important operation we study in this chapter is the Fourier transform, the father of all oscillatory integrals. This is as fundamental to Fourier analysis as marrow is to the human bone. It is a powerful transformation that carries a function from its spatial domain to its frequency domain. By doing this, it inverts the function's localization properties. If applied one more time, then magically reproduces the function composed with a reflection. It changes convolution to multiplication, translation to modulation, and expanding dilation to shrinking dilation. Its decay at infinity encodes information about the local smoothness of the function. The study of the Fourier transform also motivates the launch of a thorough study of general oscillatory integrals. We take a quick look at this topic with emphasis on one-dimensional results.

Distributions supply a mathematical framework for many operations that do not exactly qualify to be called functions. These operations found their mathematical place in the world of functionals applied to smooth functions (called test functions). These functionals also introduced the correct interpretation for many physical objects, such as the Dirac delta function. Distributions have become an indispensable tool in analysis and have enhanced our perspective.