Consider the sequence $\{g_m\}_{m=1}^{\infty} = \{f_{1,1}, f_{2,1}, f_{2,2}, f_{3,1}, f_{3,2}, f_{3,3}, \dots\}$. Observe that

$$|\{x:\ f_{k,j}(x) > 0\}| = 1/k, \qquad 1 \le j \le k.$$

Therefore, g_m converges to 0 in measure as $m \to \infty$. Likewise, observe that

$$\big\|f_{k,k}\big\|_{L^{p,\infty}} = \sup_{\alpha>0} \alpha|\{x:\ f_{k,k}(x) > \alpha\}|^{1/p} \ge \frac{(k-1/k)^{1/p}}{k^{1/p}} \to 1\,, \quad \text{as } k \to \infty,$$

which implies that $f_{k,k}$ does not converge to 0 in $L^{p,\infty}$, hence so does g_m.

It turns out that every sequence convergent in $L^p(X,\mu)$ or in $L^{p,\infty}(X,\mu)$ has a subsequence that converges a.e. to the same limit.

Theorem 1.1.11. *Let f_n and f be complex-valued measurable functions on a measure space (X,μ) and suppose that f_n converges to f in measure. Then some subsequence of f_n converges to f μ-a.e.*

Proof. For all $k = 1, 2, \dots$ choose inductively n_k such that

$$\mu(\{x \in X:\ |f_{n_k}(x) - f(x)| > 2^{-k}\}) < 2^{-k} \tag{1.1.16}$$

and such that $n_1 < n_2 < \cdots < n_k < \cdots$. Define the sets

$$A_k = \{x \in X:\ |f_{n_k}(x) - f(x)| > 2^{-k}\}\,.$$

Equation (1.1.16) implies that

$$\mu\Big(\bigcup_{k=m}^{\infty} A_k\Big) \le \sum_{k=m}^{\infty} \mu(A_k) \le \sum_{k=m}^{\infty} 2^{-k} = 2^{1-m} \tag{1.1.17}$$

for all $m = 1, 2, 3, \dots$. It follows from (1.1.17) that

$$\mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) \le 1 < \infty\,. \tag{1.1.18}$$

Using (1.1.17) and (1.1.18), we conclude that the sequence of the measures of the sets $\{\bigcup_{k=m}^{\infty} A_k\}_{m=1}^{\infty}$ converges as $m \to \infty$ to

$$\mu\Big(\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k\Big) = 0\,. \tag{1.1.19}$$

To finish the proof, observe that the null set in (1.1.19) contains the set of all $x \in X$ for which $f_{n_k}(x)$ does not converge to $f(x)$. □

In many situations we are given a sequence of functions and we would like to extract a convergent subsequence. One way to achieve this is via the next theorem, which is a useful variant of Theorem 1.1.11. We first give a relevant definition.