satisfies $\alpha|\{x: \ |S(\chi_A)(x)|>\alpha\}|^{1/q} \le c|A|^{1/2}$ when $q=1$ or 3, and thus it furnishes an example of an operator of restricted weak types $(2,1)$ and $(2,3)$ that is not L^2 bounded. Thus Theorem 1.4.19 fails if the assumption $p_0 \neq p_1$ is dropped.

Next, we give a corollary of the proof of Theorem 1.4.19 which also strengthens Theorem 1.3.2.

Corollary 1.4.24. *Let $0<r\le\infty$, $0<p_0\neq p_1\le\infty$, and $0<q_0\neq q_1\le\infty$ and let (X,μ) and (Y,ν) be σ-finite measure spaces. Let T be a quasi-linear operator defined on $L^{p_0,m}(X)+L^{p_1,m}(X)$ and taking values in the set of measurable functions on Y. Let $0<m<\infty$. Assume that for some $M_0',M_1'<\infty$ the estimates hold for $j=0,1$*

$$\big\|T(f)\big\|_{L^{q_j,\infty}(Y)} \le M_j'\big\|f\big\|_{L^{p_j,m}(X)}, \tag{1.4.42}$$

for all functions $f\in L^{p_j,m}(X)$. Fix $0<\theta<1$ and let

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1} \qquad \text{and} \qquad \frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}. \tag{1.4.43}$$

Then there exists a constant $C_*(p_0,q_0,p_1,q_1,K,r,\theta,m)<\infty$ such that for all functions f in $L^{p,r}(X)$ we have

$$\big\|T(f)\big\|_{L^{q,r}} \le C_*(p_0,q_0,p_1,q_1,K,r,\theta,m)(M_0')^{1-\theta}(M_1')^{\theta}\big\|f\big\|_{L^{p,r}}. \tag{1.4.44}$$

Proof. Since $L^{p,r}(X)$ is contained in $L^{p_0,m}(X)+L^{p_1,m}(X)$, the operator T is well defined on $L^{p,r}(X)$. Hypothesis (1.4.42) implies that (1.4.30) holds for all f in $L^{p_0,m}$ and $L^{p_1,m}$, respectively. We obtain (1.4.44) by repeating the proof of Theorem 1.4.19 starting from (1.4.30) and working a fixed function f in $L^{p,r}(X)$. Note that the restriction $p_1<\infty$ was never used in the proof of Theorem 1.4.19, and was only needed to apply Lemma 1.4.20, which is not necessary in this setting. □

Theorem 1.4.25. *(**Young's inequality for weak type spaces**) Let G be a locally compact group with left Haar measure λ that satisfies (1.2.12) for all measurable subsets A of G. Let $1<p,q,r<\infty$ satisfy*

$$\frac{1}{q}+1=\frac{1}{p}+\frac{1}{r}. \tag{1.4.45}$$

Then there exists a constant $B_{p,q,r}>0$ such that for all f in $L^p(G)$ and g in $L^{r,\infty}(G)$ we have

$$\big\|f*g\big\|_{L^q(G)} \le B_{p,q,r}\big\|g\big\|_{L^{r,\infty}(G)}\big\|f\big\|_{L^p(G)}. \tag{1.4.46}$$

Proof. We fix $1<p,q<\infty$. Since p and q range in an open interval, we can find $p_0<p<p_1$, $q_0<q<q_1$, and $0<\theta<1$ such that (1.4.23) and (1.4.45) hold. Let $T(f)=f*g$, defined for all functions f on G. By Theorem 1.2.13, T extends to a bounded operator from L^{p_0} to $L^{q_0,\infty}$ and from L^{p_1} to $L^{q_1,\infty}$. It follows from the Corollary 1.4.24 that T extends to a bounded operator from $L^p(G)$ to $L^q(G)$. Notice that since G is locally compact, (G,λ) is a σ-finite measure space and for this reason, we were able to apply Corollary 1.4.24. □