on $S_0(X)$ and $\overline{T}$ is bounded from $L^{p,r}(X)$ to $L^{q,r}(Y)$. Thus $\overline{T}$ is the unique bounded extension of T on the entire space $L^{p,r}(X)$. For details, see Exercise 1.4.17. $\square$

Proposition 1.4.21. *For all* $0 < p, r < \infty$ *the space* $S_0(X)$ *is dense in* $L^{p,r}(X)$.

Proof. Let $f \in L^{p,r}(X)$ and assume first that $f \geq 0$. Using (1.4.5) and the fact that d_f is decreasing on $[0, \infty)$, we obtain for any $n \in \mathbf{Z}^+$,

$$\begin{aligned} \|f\|_{L^{p,r}(X)}^r &= p \int_0^\infty \left[d_f(s)^{\frac{1}{p}} s \right]^r \frac{ds}{s} \\ &\geq p \int_0^{2^{-n}} \left[d_f(2^{-n}) \right]^{\frac{r}{p}} s^{r-1} \, ds \\ &= \frac{p 2^{-nr}}{r} \left[d_f(2^{-n}) \right]^{\frac{r}{p}}, \end{aligned}$$

which implies that $d_f(2^{-n}) < \infty$. Likewise, again in view of (1.4.5), we have

$$\|f\|_{L^{p,r}(X)}^r \geq p \int_0^{2^n} \left[d_f(s) \right]^{\frac{r}{p}} s^{r-1} \, ds \geq \frac{p 2^{nr}}{r} \left[d_f(2^n) \right]^{\frac{r}{p}},$$

which implies that $\lim_{n\to\infty} d_f(2^n) = 0$. Thus, for any $n \in \mathbf{Z}^+$, there exists $k_n \in \mathbb{N}$ such that

$$d_f(2^{k_n}) = \mu\left(\left\{x \in X : f(x) > 2^{k_n}\right\}\right) < 2^{-n}.$$

Let $E_n = \left\{x \in X : 2^{-n} < f(x) \leq 2^{k_n}\right\}$ and note that $\mu\left(E_n\right) \leq d_f\left(2^{-n}\right) < \infty$ for each $n \in \mathbf{Z}^+$. We write $f\chi_{E_n}$ in binary expansion, that is, $f\chi_{E_n}(x) = \sum_{j=-k_n}^{\infty} d_j(x) 2^{-j}$, where $d_j(x) = 0$ or 1. Let $B_j = \{x \in E_n : d_j(x) = 1\}$. Then, $\mu(B_j) \leq \mu(E_n)$ and $f\chi_{E_n}$ can be expressed as $f\chi_{E_n} = \sum_{j=-k_n}^{\infty} 2^{-j} \chi_{B_j}$.

Set $f_n = \sum_{j=-k_n}^{n} 2^{-j} \chi_{B_j}$. It is obvious that $f_n \in S_0^+(X)$ and $f_n \leq f\chi_{E_n} \leq f$. Observe that when $x \in E_n$, we have

$$f(x) - f_n(x) = \sum_{j=n+1}^{\infty} 2^{-j} \chi_{B_j} \leq 2^{-n},$$

and that when $x \notin E_n$, we have $f_n(x) = 0$ and $f(x) > 2^{k_n}$ or $f(x) \leq 2^{-n}$. It follows from these facts that

$$d_{f-f_n}(2^{-n}) = \mu\left(E_n \cap \{f - f_n > 2^{-n}\}\right) + \mu\left(E_n^c \cap \{f - f_n > 2^{-n}\}\right) < 2^{-n}.$$

Hence, for $2^{-n} \leq t < \infty$ one has

$$(f - f_n)^*(t) \leq (f - f_n)^*(2^{-n}) = \inf\left\{s > 0 : d_{f-f_n}(s) \leq 2^{-n}\right\} \leq 2^{-n}.$$

This implies that $\lim_{n\to\infty}(f - f_n)^*(t) = 0$ for all $t \in (0, \infty)$. By Proposition 1.4.5 (4), (6), we obtain for all $t \in (0, \infty)$

$$(f - f_n)^*(t) \leq f^*(t/2) + f_n^*(t/2) \leq 2f^*(t/2).$$