to estimate (1.4.31) by

$$K\max\{1,2^{\frac{1}{r}-1}\}\left[2^{\frac{1}{q_0}}M_0'\left\|t^{\frac{1}{q}-\frac{1}{q_0}}\|f^t\|_{L^{p_0,m}}\right\|_{L^r(\frac{dt}{t})}+2^{\frac{1}{q_1}}M_1'\left\|t^{\frac{1}{q}-\frac{1}{q_1}}\|f_t\|_{L^{p_1,m}}\right\|_{L^r(\frac{dt}{t})}\right],$$

which is the same as

$$K\max\{1,2^{\frac{1}{r}-1}\}2^{\frac{1}{q_0}}M_0'\left\|t^{-\gamma(\frac{1}{p_0}-\frac{1}{p})}\|f^t\|_{L^{p_0,m}}\right\|_{L^r(\frac{dt}{t})} \tag{1.4.34}$$

$$+K\max\{1,2^{\frac{1}{r}-1}\}2^{\frac{1}{q_1}}M_1'\left\|t^{\gamma(\frac{1}{p}-\frac{1}{p_1})}\|f_t\|_{L^{p_1,m}}\right\|_{L^r(\frac{dt}{t})}. \tag{1.4.35}$$

Next, we change variables $u=\delta t^\gamma$ in the L^r quasi-norm in (1.4.34) to obtain

$$\begin{aligned}&\left\|t^{-\gamma(\frac{1}{p_0}-\frac{1}{p})}\|f^t\|_{L^{p_0,m}}\right\|_{L^r(\frac{dt}{t})}\\&\quad\le\frac{\delta^{\frac{1}{p_0}-\frac{1}{p}}}{|\gamma|^{\frac{1}{r}}}\left\|u^{-(\frac{1}{p_0}-\frac{1}{p})}\left(\int_0^u f^*(s)^m s^{\frac{m}{p_0}}\frac{ds}{s}\right)^{\frac{1}{m}}\right\|_{L^r(\frac{du}{u})}\\&\quad\le\frac{\delta^{\frac{1}{p_0}-\frac{1}{p}}}{|\gamma|^{\frac{1}{r}}}\left[\frac{\frac{r}{m}}{r(\frac{1}{p_0}-\frac{1}{p})}\right]^{\frac{1}{m}}\left(\int_0^\infty(s^{\frac{1}{p_0}}f^*(s))^r s^{-r(\frac{1}{p_0}-\frac{1}{p})}\frac{ds}{s}\right)^{\frac{1}{r}}\\&\quad=\frac{\delta^{\frac{1}{p_0}-\frac{1}{p}}}{m^{\frac{1}{m}}|\gamma|^{\frac{1}{r}}(\frac{1}{p_0}-\frac{1}{p})^{\frac{1}{m}}}\|f\|_{L^{p,r}},\end{aligned}$$

where the last inequality is a consequence of Hardy's inequality:

$$\left(\int_0^\infty\left(\int_0^u g(s)\frac{ds}{s}\right)^p u^{-b}\frac{du}{u}\right)^{\frac{1}{p}}\le\frac{p}{b}\left(\int_0^\infty g(u)^p u^{-b}\frac{du}{u}\right)^{\frac{1}{p}} \tag{1.4.36}$$

with $g(s)=f^*(s)^m s^{m/p_0}\ge 0$, $p=r/m\ge 1$ and $b=r/p_0-r/p>0$. See Exercise 1.2.8 for the proof of (1.4.36).

Likewise, change variables $u=\delta t^\gamma$ in the L^r quasi-norm of (1.4.35) to obtain

$$\begin{aligned}&\left\|t^{\gamma(\frac{1}{p}-\frac{1}{p_1})}\|f_t\|_{L^{p_1,m}}\right\|_{L^r(\frac{dt}{t})}\\&\le\frac{\delta^{-(\frac{1}{p}-\frac{1}{p_1})}}{|\gamma|^{\frac{1}{r}}}\left\|u^{\frac{1}{p}-\frac{1}{p_1}}\left[\int_0^u f^*(u)^m s^{\frac{m}{p_1}}\frac{ds}{s}+\int_u^\infty f^*(s)^m s^{\frac{m}{p_1}}\frac{ds}{s}\right]^{\frac{1}{m}}\right\|_{L^r(\frac{du}{u})}\\&=\frac{\delta^{-(\frac{1}{p}-\frac{1}{p_1})}}{|\gamma|^{\frac{1}{r}}}\left\|u^{\frac{m}{p}-\frac{m}{p_1}}\int_0^u f^*(u)^m s^{\frac{m}{p_1}}\frac{ds}{s}+u^{\frac{m}{p}-\frac{m}{p_1}}\int_u^\infty f^*(s)^m s^{\frac{m}{p_1}}\frac{ds}{s}\right\|_{L^{r/m}(\frac{du}{u})}^{\frac{1}{m}}\end{aligned}$$