Additionally, if $0 < p, r < \infty$ and if T is linear (or sublinear with nonnegative values), then it admits a unique bounded extension from $L^{p,r}(X)$ to $L^{q,r}(Y,\nu)$ such that (1.4.24) holds for all f in $L^{p,r}$.

Before we give the proof of Theorem 1.4.19, we state and prove a lemma that is interesting on its own.

Lemma 1.4.20. *Let $0 < p < \infty$ and $0 < q \le \infty$ and let (X,μ), (Y,ν) be σ-finite measure spaces. Let T be a quasi-linear operator defined on $S(X)$ and taking values in the set of measurable functions on Y. Suppose that there exists a constant $M > 0$ such that for all measurable subsets A of X of finite measure we have*

$$\big\|T(\chi_A)\big\|_{L^{q,\infty}} \le M\,\mu(A)^{\frac{1}{p}}. \tag{1.4.25}$$

Then for all α with $0 < \alpha < \min(q, \frac{\log 2}{\log 2K})$ there exists a constant $C(p,q,K,\alpha) > 0$ such that for all functions f in $S_0(X)$ we have the estimate

$$\big\|T(f)\big\|_{L^{q,\infty}} \le C(p,q,K,\alpha)M\,\big\|f\big\|_{L^{p,\alpha}} \tag{1.4.26}$$

where

$$C(p,q,K,\alpha) = 2^{8+\frac{2}{p}+\frac{2}{q}}K^3\left(\frac{q}{q-\alpha}\right)^{\frac{2}{\alpha}}(1-2^{-\alpha})^{-\frac{1}{\alpha}}(\log 2)^{-\frac{1}{\alpha}}.$$

Proof. A function f in $S_0(X)$ can be written as $f = h_1 - h_2 + i(h_3 - h_4)$, where h_j are in $S_0^+(X)$. We write $f = f_1 - f_2 + i(f_3 - f_4)$, where $f_1 = \max(h_1 - h_2, 0)$, $f_2 = \max(-(h_1 - h_2), 0)$, $f_3 = \max(h_3 - h_4, 0)$, and $f_4 = \max(-(h_3 - h_4), 0)$. We note that f_j lie in $S_0^+(X)$; indeed, if $h_1 = \sum_\ell 2^{-\ell}\chi_{A_\ell}$ and $h_2 = \sum_k 2^{-k}\chi_{B_k}$, where both sums are finite, then

$$f_1 = \sum_{\ell:\, A_\ell \cap (\cup_k B_k) = \emptyset} 2^{-\ell}\chi_{A_\ell} + \sum_{(\ell,k):\, \ell<k,\, A_\ell\cap B_k \neq \emptyset} (2^{-\ell} - 2^{-k})\chi_{A_\ell\cap B_k}.$$

Since the second sum is equal to $\sum_{s=\ell+1}^{k} 2^{-s}\chi_{A_\ell\cap B_k}$, we obtain that $f_1 \in S_0^+(X)$. Likewise we can show that f_2, f_3, f_4 lie in $S_0^+(X)$. Moreover, we have $f_j \le |f|$ and Proposition 1.4.5(4), yields

$$\|f_j\|_{L^{p,\alpha}(X)} \le \|f\|_{L^{p,\alpha}(X)}$$

for all $j = 1,2,3,4$. Suppose now that (1.4.26) holds for functions in $S_0^+(X)$ with constant $C'(p,q,\alpha)$ in place of $C(p,q,K,\alpha)$. By the quasi-linearity of T we have