Appendix I
Taylor's and Mean Value Theorem in Several Variables

I.1 Mutlivariable Taylor's Theorem

For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbf{Z}^+ \cup \{0\})^n$, we denote by $|\alpha| = \alpha_1 + \cdots + \alpha_n$ its size, we define $\alpha! = \alpha_1! \cdots \alpha_n!$ its factorial, and we set

$$h^\alpha = h_1^{\alpha_1} \cdots h_n^{\alpha_n},$$

where $h = (h_1, \ldots, h_n)$; here $0^0 = 1$.

Let $k \in \mathbf{Z}^+ \cup \{0\}$. Suppose a real-valued $\mathscr{C}^{k+1}$ function f is defined on an open convex subset Ω of $\mathbf{R}^n$. Suppose that $x \in \Omega$ and $x + h \in \Omega$. Then we have the *Taylor expansion formula*

$$f(x+h) = \sum_{|\alpha| \le k} \frac{\partial^\alpha f(x)}{\alpha!} h^\alpha + R(h,x,k),$$

where the remainder $R(h,x,k)$ can be expressed in integral form

$$R(h,x,k) = (k+1) \sum_{|\alpha|=k+1} \frac{h^\alpha}{\alpha!} \int_0^1 (1-t)^k \partial^\alpha f(x+th)\, dt.$$

If in addition f is real-valued, then the remainder can be expressed in Langrange's mean value form

$$R(h,x,k) = \sum_{|\alpha|=k+1} \frac{\partial^\alpha f(x+ch)}{\alpha!} h^\alpha$$

for some $c \in (0,1)$.

9.2 The Mean value Theorem

Suppose that f is as above and $k = 0$. Then for given $x, y \in \Omega$ we have

$$f(y) - f(x) = \int_0^1 \nabla f((1-t)x + ty) \cdot (y - x)\, dt$$

and moreover, if f is real-valued, then

$$f(y) - f(x) = \nabla f((1-c)x + cy) \cdot (y - x)$$

for some $c \in (0,1)$. This is a special case of Taylor's formula when $k = 0$.

Chapter 10
The Whitney Decomposition of Open Sets in $\mathbf{R}^n$

10.1 Decomposition of Open Sets

An arbitrary open set in $\mathbf{R}^n$ can be decomposed as a union of disjoint cubes whose lengths are proportional to their distance from the boundary of the open set. See, for instance, Figure 10.1 when the open set is the unit disk in $\mathbf{R}^2$. For a given cube Q in $\mathbf{R}^n$, we denote by $\ell(Q)$ its length.

Proposition. *Let Ω be an open nonempty proper subset of $\mathbf{R}^n$. Then there exists a family of closed dyadic cubes $\{Q_j\}_j$ (called the Whitney cubes of Ω) such that*
(a) $\bigcup_j Q_j = \Omega$ and the Q_j's have disjoint interiors.
(b) $\sqrt{n}\,\ell(Q_j) \le \operatorname{dist}(Q_j, \Omega^c) \le 4\sqrt{n}\,\ell(Q_j)$. Thus $10\sqrt{n}\,Q_j$ meets Ω^c.
(c) If the boundaries of two cubes Q_j and Q_k touch, then

$$\frac{1}{4} \le \frac{\ell(Q_j)}{\ell(Q_k)} \le 4\,.$$

(d) For a given Q_j there exist at most $12^n - 4^n$ cubes Q_k that touch it.
(e) Let $0 < \varepsilon < 1/4$. If Q_j^* has the same center as Q_j and $\ell(Q_j^*) = (1+\varepsilon)\ell(Q_j)$ then

$$\chi_\Omega \le \sum_j \chi_{Q_j^*} \le 12^n - 4^n + 1\,.$$

Proof. Let $\mathscr{D}_k$ be the collection of all dyadic cubes of the form

$$\{(x_1, \dots, x_n) \in \mathbf{R}^n : \ m_j 2^{-k} \le x_j < (m_j + 1)2^{-k}\}\,,$$

where $m_j \in \mathbf{Z}$. Observe that each cube in $\mathscr{D}_k$ gives rise to 2^n cubes in $\mathscr{D}_{k+1}$ by bisecting each side.

Write the set Ω as the union of the sets

$$\Omega_k = \{x \in \Omega : \ 2\sqrt{n}\,2^{-k} < \operatorname{dist}(x, \Omega^c) \le 4\sqrt{n}\,2^{-k}\}$$

over all $k \in \mathbf{Z}$. Let $\mathscr{F}'$ be the set of all cubes Q in $\mathscr{D}_k$ for some $k \in \mathbf{Z}$ such that $Q \cap \Omega_k \neq \emptyset$. We show that the collection $\mathscr{F}'$ satisfies property (b). Let $Q \in \mathscr{F}'$ and pick $k \in \mathbf{Z}$ such that $Q \in \mathscr{D}_k$ and $x \in \Omega_k \cap Q$. Observe that

$$\sqrt{n}\,2^{-k} \leq \operatorname{dist}(x, \Omega^c) - \sqrt{n}\,\ell(Q) \leq \operatorname{dist}(Q, \Omega^c) \leq \operatorname{dist}(x, \Omega^c) \leq 4\sqrt{n}\,2^{-k},$$

which proves (b).

Next we observe that

$$\bigcup_{Q \in \mathscr{F}'} Q = \Omega.$$

Indeed, every Q in $\mathscr{F}'$ is contained in Ω (since it has positive distance from its complement) and every $x \in \Omega$ lies in some Ω_k and in some dyadic cube in $\mathscr{D}_k$.

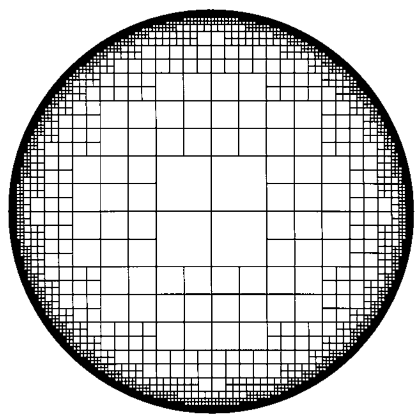

Fig. 10.1 The Whitney decomposition of the unit disk.

The problem is that the cubes in the collection $\mathscr{F}'$ may not be disjoint. We have to refine the collection $\mathscr{F}'$ by eliminating those cubes that are contained in some other cubes in the collection. Recall that two dyadic cubes have disjoint interiors or else one contains the other. For every cube Q in $\mathscr{F}'$ we can therefore consider the unique *maximal* cube $Q^{\max}$ in $\mathscr{F}'$ that contains it. Two different such maximal cubes must have disjoint interiors by maximality. Now set $\mathscr{F} = \{Q^{\max} : Q \in \mathscr{F}'\}$.

The collection of cubes $\{Q_j\}_j = \mathscr{F}$ clearly satisfies (a) and (b), and we now turn our attention to the proof of (c). Observe that if Q_j and Q_k in $\mathscr{F}$ touch then

$$\sqrt{n}\,\ell(Q_j) \leq \operatorname{dist}(Q_j, \Omega^c) \leq \operatorname{dist}(Q_j, Q_k) + \operatorname{dist}(Q_k, \Omega^c) \leq 0 + 4\sqrt{n}\,\ell(Q_k),$$

which proves (c). To prove (d), note that any cube Q in $\mathscr{D}_k$ is touched by exactly $3^n - 1$ other cubes in $\mathscr{D}_k$. But each cube Q in $\mathscr{D}_k$ can contain at most 4^n cubes of $\mathscr{F}$ of length at least one-quarter of the length of Q. This fact combined with (c) yields (d). To prove (e), notice that each Q_j^* is contained in Ω by part (b). If $x \in \Omega$, then