Appendix G
Basic Functional Analysis

A quasi-norm is a nonnegative functional $\|\cdot\|$ on a vector space X that satisfies $\|x+y\|_X \le K(\|x\|_X + \|y\|_X)$ for some $K \ge 0$ and all $x, y \in X$ and also $\|\lambda x\|_X = |\lambda|\|x\|_X$ for all scalars λ and $\|x\|_X = 0 \implies x = 0$. When $K = 1$, the quasi-norm is called a norm. A quasi-Banach space is a quasi-normed space that is complete with respect to the topology generated by the quasi-norm. The proofs of the following theorems can be found in several books including Albiac and Kalton [1], Kalton, Peck, and Roberts [188], and Rudin [306].

The Hahn–Banach theorem. Let X be a normed vector space (over the real or complex numbers), let Y be a subspace of X, and let P be a positively homogeneous subadditive[1] functional on X. Then for every linear functional Λ on Y that satisfies

$$|\Lambda(y)| \le P(y)$$

for all $y \in Y$, there is a linear functional Λ' on X such that

$$\begin{aligned} \Lambda'(y) &= \Lambda(y) \qquad \text{for all } y \in Y, \\ |\Lambda'(x)| &\le P(x) \qquad \text{for all } x \in X. \end{aligned}$$

In particular, every bounded linear functional on a subspace has an extension on the entire space with the same norm.

Banach–Alaoglou theorem. Let X be a normed vector space and let X^* be the space of all bounded linear functionals on X. Then the closed unit ball of X^* is compact in the weak* topology.

A special case is the sequential version of this theorem, which asserts that the closed unit ball of the dual space of a separable normed vector space is sequentially compact in the weak* topology. Indeed, the weak* topology on the closed unit ball

[1] this means that $P(x) \ge 0$ for all $x \in X$, $P(\lambda x) = \lambda P(x)$ for all $\lambda > 0$ and all $x \in X$, and that $P(x+z) \le P(x) + P(z)$ for all $x, z \in X$.