noting that the preceding integral converges since $(g_{N,M})^*(s) \le M\chi_{[0,\mu(K_N))]}(s)$. It follows that $f^*(t) \le c_{p,q} M^{q'-1}[\mu(K_N)^{q'/p'-1} + (t/2)^{q'/p'-1} + \log\frac{2\mu(K_N)}{t}]$ and also $f^*(t) = 0$ when $t > 2\mu(K_N)$, thus the function f defined in (1.4.16) lies in $L^{p,q}(X)$.

We have the following calculation regarding the $L^{p,q}$ norm of f:

$$\begin{aligned} \|f\|_{L^{p,q}} &= \left(\int_0^\infty t^{\frac{q}{p}} \left[\int_{t/2}^\infty s^{\frac{q'}{p'}-1}(g_{N,M})^*(s)^{q'-1}\frac{ds}{s}\right]^q \frac{dt}{t}\right)^{\frac{1}{q}} \\ &\le C_1(p,q)\left(\int_0^\infty (t^{\frac{1}{p'}}(g_{N,M})^*(t))^{q'}\frac{dt}{t}\right)^{\frac{1}{q}} \\ &= C_1(p,q)\|g_{N,M}\|_{L^{p',q'}}^{q'/q} < \infty, \end{aligned} \tag{1.4.17}$$

which is a consequence of Hardy's second inequality in Exercise 1.2.8 with $b = q/p$.

Using (1.4.15) and (1.4.17) we deduce that

$$\int_0^\infty f^*(t)(g_{N,M})^*(t)\,dt \le \|T\|\,\|f\|_{L^{p,q}} \le C_1(p,q)\|T\|\,\|g_{N,M}\|_{L^{p',q'}}^{q'-1}. \tag{1.4.18}$$

On the other hand, we have

$$\begin{aligned} \int_0^\infty f^*(t)(g_{N,M})^*(t)\,dt &\ge \int_0^\infty \int_{t/2}^t s^{\frac{q'}{p'}-1}(g_{N,M})^*(s)^{q'-1}\frac{ds}{s}(g_{N,M})^*(t)\,dt \\ &\ge \int_0^\infty (g_{N,M})^*(t)^{q'}\int_{t/2}^t s^{\frac{q'}{p'}-1}\frac{ds}{s}\,dt \\ &= C_2(p,q)\|g_{N,M}\|_{L^{p',q'}}^{q'}. \end{aligned} \tag{1.4.19}$$

Combining (1.4.18) and (1.4.19), and using the fact that $\|g_{N,M}\|_{L^{p',q'}} < \infty$, we obtain $\|g_{N,M}\|_{L^{p',q'}} \le C(p,q)\|T\|$. Letting $N,M \to \infty$ we deduce $\|g\|_{L^{p',q'}} \le C(p,q)\|T\|$ and this proves the reverse inequality required to complete case (vi).

(vii) For a complete characterization of this space, we refer to [83].

(viii) The dual of $L^\infty = L^{\infty,\infty}$ can be identified with the set of all bounded finitely additive set functions; see [99]. □

Remark 1.4.17. Some parts of Theorem 1.4.16 are false if X is atomic. For instance, the dual of $\ell^p(\mathbf{Z})$ contains ℓ^∞ when $0 < p < 1$ and thus it is not equal to $\{0\}$.

1.4.4 The Off-Diagonal Marcinkiewicz Interpolation Theorem

We now present the main result of this section, the off-diagonal extension of Marcinkiewicz's interpolation theorem (Theorem 1.3.2). For a measure space (X,μ), let $S(X)$ be the space of finitely simple functions on X and $S_0^+(X)$ be the subset of $S(X)$ of functions of the form