where we used the inequality $\frac{1}{2}(e^x+e^{-x}) \le e^{\frac{1}{2}x^2}$ for all real x, which can be checked using power series expansions. Since the same argument is also valid for $-\sum a_j r_j(t)$, we obtain that

$$\int_0^1 e^{\rho|F(t)|}\,dt \le 2e^{\frac{1}{2}\rho^2}.$$

From this it follows that

$$e^{\rho\alpha}|\{t\in[0,1]:\ |F(t)|>\alpha\}| \le \int_0^1 e^{\rho|F(t)|}\,dt \le 2e^{\frac{1}{2}\rho^2}$$

and hence we obtain the distributional inequality

$$d_F(\alpha) = |\{t\in[0,1]:\ |F(t)|>\alpha\}| \le 2e^{\frac{1}{2}\rho^2-\rho\alpha} = 2e^{-\frac{1}{2}\alpha^2},$$

by picking $\rho=\alpha$. The L^p norm of F can now be computed easily. Formula (1.1.6) gives

$$\big\|F\big\|_{L^p}^p = \int_0^\infty p\alpha^{p-1} d_F(\alpha)\,d\alpha \le \int_0^\infty p\alpha^{p-1} 2e^{-\frac{\alpha^2}{2}}\,d\alpha = 2^{\frac{p}{2}}p\Gamma(p/2).$$

We have now proved that

$$\big\|F\big\|_{L^p} \le \sqrt{2}\,\big(p\Gamma(p/2)\big)^{\frac{1}{p}}\big\|F\big\|_{L^2}$$

under assumptions (a), (b), and (c).

We now dispose of assumptions (a), (b), and (c). Assumption (b) can be easily eliminated by a limiting argument and (c) by a scaling argument. To dispose of assumption (a), let a_j and b_j be real numbers. For $p>2$ we have

$$\begin{aligned}
\Big\|\sum_j (a_j+ib_j)r_j\Big\|_{L^p} &\le \Big\|\,\big|\sum_j a_jr_j\big| + \big|\sum_j b_jr_j\big|\,\Big\|_{L^p}\\
&\le \Big\|\sum_j a_jr_j\Big\|_{L^p} + \Big\|\sum_j b_jr_j\Big\|_{L^p}\\
&\le \sqrt{2}\,\big(p\Gamma(p/2)\big)^{\frac{1}{p}}\Big(\Big(\sum_j |a_j|^2\Big)^{\frac{1}{2}} + \Big(\sum_j |b_j|^2\Big)^{\frac{1}{2}}\Big)\\
&\le \sqrt{2}\,\big(p\Gamma(p/2)\big)^{\frac{1}{p}}\sqrt{2}\Big(\sum_j |a_j+ib_j|^2\Big)^{\frac{1}{2}}.
\end{aligned}$$

Let us now set $A_p = 2\big(p\Gamma(p/2)\big)^{1/p}$ when $p>2$. Since we have the trivial estimate $\big\|F\big\|_{L^p} \le \big\|F\big\|_{L^2}$ when $0<p\le 2$, we obtain the required inequality $\big\|F\big\|_{L^p} \le A_p\big\|F\big\|_{L^2}$ with

$$A_p = \begin{cases} 1 & \text{when } 0<p\le 2,\\ 2\,p^{\frac{1}{p}}\,\Gamma(p/2)^{\frac{1}{p}} & \text{when } 2<p<\infty. \end{cases}$$