Estimating the two integrals on the right by putting absolute values inside and multiplying by the missing factor $r^\nu 2^{-\nu}(\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2}))^{-1}$, we obtain

$$|J_\nu(r)| \le 2\,\frac{(r/2)^{\operatorname{Re}\nu}\,e^{\frac{\pi}{2}|\operatorname{Im}\nu|}}{|\Gamma(\nu+\frac{1}{2})|\,\Gamma(\frac{1}{2})}\int_0^\infty e^{-rt}t^{\operatorname{Re}\nu-\frac{1}{2}}\left(\sqrt{t^2+4}\right)^{\operatorname{Re}\nu-\frac{1}{2}}\,dt\,,$$

since the absolute value of the argument of $t^2\pm 2it$ is at most $\pi/2$. When $\operatorname{Re}\nu>1/2$, we use the inequality $(\sqrt{t^2+4})^{\operatorname{Re}\nu-\frac{1}{2}} \le 2^{\operatorname{Re}\nu-\frac{1}{2}}\left(t^{\operatorname{Re}\nu-\frac{1}{2}}+2^{\operatorname{Re}\nu-\frac{1}{2}}\right)$ to get

$$|J_\nu(r)| \le 2\,\frac{(r/2)^{\operatorname{Re}\nu}\,e^{\frac{\pi}{2}|\operatorname{Im}\nu|}}{|\Gamma(\nu+\frac{1}{2})|\,\Gamma(\frac{1}{2})}\,2^{\operatorname{Re}\nu-\frac{1}{2}}\left[\frac{\Gamma(2\operatorname{Re}\nu)}{r^{2\operatorname{Re}\nu}}+2^{\operatorname{Re}\nu}\frac{\Gamma(\operatorname{Re}\nu+\frac{1}{2})}{r^{\operatorname{Re}\nu+\frac{1}{2}}}\right].$$

When $1/2\ge \operatorname{Re}\nu>-1/2$ we use that $\left(\sqrt{t^2+4}\right)^{\operatorname{Re}\nu-\frac{1}{2}}\le 1$ to deduce that

$$|J_\nu(r)| \le 2\,\frac{(r/2)^{\operatorname{Re}\nu}\,e^{\frac{\pi}{2}|\operatorname{Im}\nu|}}{|\Gamma(\nu+\frac{1}{2})|\,\Gamma(\frac{1}{2})}\,\frac{\Gamma(\operatorname{Re}\nu+\frac{1}{2})}{r^{\operatorname{Re}\nu+\frac{1}{2}}}\,.$$

These estimates yield that for $\operatorname{Re}\nu>-1/2$ and $r\ge 1$ we have

$$|J_\nu(r)| \le C_1(\operatorname{Re}\nu)\,e^{\left(\max\left((\operatorname{Re}\nu+\frac{1}{2})^{-2},(\operatorname{Re}\nu+\frac{1}{2})^{-1}\right)+\frac{\pi}{2}\right)|\operatorname{Im}\nu|^2}\,r^{-1/2}$$

using the result in Appendix A.7, where C_1 is a constant that depends smoothly on $\operatorname{Re}\nu$ on the interval $(-1/2,\infty)$.

B.8 Asymptotics of Bessel Functions

We obtain asymptotics for $J_\nu(r)$ as $r\to\infty$ whenever $\operatorname{Re}\nu>-1/2$. We have the following identity for $r>0$:

$$J_\nu(r)=\sqrt{\frac{2}{\pi r}}\cos\left(r-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)+R_\nu(r)\,,$$

where R_ν is given by

$$\begin{aligned}R_\nu(r) \;=\; &\frac{(2\pi)^{-\frac{1}{2}}r^\nu}{\Gamma(\nu+\frac{1}{2})}e^{i(r-\frac{\pi\nu}{2}-\frac{\pi}{4})}\int_0^\infty e^{-rt}t^{\nu+\frac{1}{2}}\left[(1+\tfrac{it}{2})^{\nu-\frac{1}{2}}-1\right]\frac{dt}{t}\\ &+\frac{(2\pi)^{-\frac{1}{2}}r^\nu}{\Gamma(\nu+\frac{1}{2})}e^{-i(r-\frac{\pi\nu}{2}-\frac{\pi}{4})}\int_0^\infty e^{-rt}t^{\nu+\frac{1}{2}}\left[(1-\tfrac{it}{2})^{\nu-\frac{1}{2}}-1\right]\frac{dt}{t}\end{aligned}$$

and satisfies $|R_\nu(r)|\le C_\nu\,r^{-3/2}$ whenever $r\ge 1$.