$$|T(f)| \le \sum_{j=1}^{m} |T(f_j)| \le \|T\| \sum_{j=1}^{m} \big\|f_j\big\|_{L^{p,q}} = \|T\|\, m^{1-1/p} \|f\|_{L^{p,q}}\,.$$

Let $m \to \infty$ and use that $p < 1$ to obtain that $T = 0$.

(ii) We now consider the case $p = 1$ and $0 < q \le 1$. Clearly, every $h \in L^\infty$ gives a bounded linear functional on $L^{1,q}$, since

$$\left|\int_X f\,h\,d\mu\right| \le \big\|h\big\|_{L^\infty}\big\|f\big\|_{L^1} \le C_q\big\|h\big\|_{L^\infty}\big\|f\big\|_{L^{1,q}}\,.$$

Conversely, suppose that $T \in (L^{1,q})^*$ where $q \le 1$. The function g given in (1.4.13) satisfies

$$\left|\int_E g\,d\mu\right| = \big|T(\chi_E)\big| \le \big\|T\big\|\, q^{-1/q}\mu(E)$$

for all $E \subseteqq K_N$, and hence $|g| \le q^{-1/q}\|T\|$ μ-a.e. on every K_N; see [307, Theorem 1.40 on p. 31] for a proof of this fact. It follows that $\|g\|_{L^\infty} \le q^{-1/q}\|T\|$ and hence $(L^{1,q})^* = L^\infty$.

(iii) Let us now take $p = 1$, $1 < q < \infty$, and suppose that $T \in (L^{1,q})^*$. Then

$$\left|\int_X f g\,d\mu\right| \le \|T\|\;\big\|f\big\|_{L^{1,q}}, \tag{1.4.14}$$

where g is the function in (1.4.13) and $f \in L^\infty(K_N)$. We will show that $g = 0$ a.e. Suppose that $|g| \ge \delta$ on some set E_0 with $\mu(E_0) > 0$. Then there exists N such that $\mu(E_0 \cap K_N) > 0$. Let $f = \overline{g}|g|^{-2}\chi_{E_0\cap K_N} h\chi_{h\le M}$, where h is a nonnegative function. Then (1.4.14) implies for all $h \ge 0$ that

$$\big\|h\chi_{h\le M}\big\|_{L^1(E_0\cap K_N)} \le \frac{1}{\delta}\|T\|\;\big\|h\chi_{h\le M}\big\|_{L^{1,q}(E_0\cap K_N)}\,.$$

Letting $M \to \infty$, we obtain that $L^{1,q}(E_0 \cap K_N)$ is contained in $L^1(E_0 \cap K_N)$, but since the reverse inclusion is always valid, these spaces must be equal. Since X is nonatomic, this can't happen; see Exercise 1.4.8 (d). Thus $g = 0$ μ-a.e. and $T = 0$.

(iv) In the case $p = 1$, $q = \infty$ an interesting phenomenon appears. Since every continuous linear functional on $L^{1,\infty}$ extends to a continuous linear functional on $L^{1,q}$ for $1 < q < \infty$, it must necessarily vanish on all simple functions by part (iii). However, $(L^{1,\infty})^*$ contains nontrivial linear functionals; see [84], [85].

(v) We now take up the case $1 < p < \infty$ and $0 < q \le 1$. Using Exercise 1.4.1 (b) and Proposition 1.4.10, we see that if $f \in L^{p,q}$ and $h \in L^{p',\infty}$, then

$$\begin{aligned}\int_X |f\,h|\,d\mu \;&\le \int_0^\infty t^{\frac{1}{p}} f^*(t) t^{\frac{1}{p'}} h^*(t)\,\frac{dt}{t}\\ &\le \big\|f\big\|_{L^{p,1}}\|h\|_{L^{p',\infty}}\\ &\le C_{p,q}\big\|f\big\|_{L^{p,q}}\|h\|_{L^{p',\infty}};\end{aligned}$$