As an application we take $f(x) = \chi_{B(0,1)}(x)$, where $B(0,1)$ is the unit ball in $\mathbf{R}^n$. We obtain

$$(\chi_{B(0,1)})\widehat{}(\xi) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_0^1 J_{\frac{n}{2}-1}(2\pi|\xi|r) r^{\frac{n}{2}}\, dr = \frac{J_{\frac{n}{2}}(2\pi|\xi|)}{|\xi|^{\frac{n}{2}}},$$

in view of the result in Appendix B.3. More generally, for $\operatorname{Re}\lambda > -1$, let

$$m_\lambda(\xi) = \begin{cases} (1-|\xi|^2)^\lambda & \text{for } |\xi| \le 1, \\ 0 & \text{for } |\xi| > 1. \end{cases}$$

Then

$$m_\lambda^\vee(x) = \frac{2\pi}{|x|^{\frac{n-2}{2}}} \int_0^1 J_{\frac{n}{2}-1}(2\pi|x|r) r^{\frac{n}{2}} (1-r^2)^\lambda\, dr = \frac{\Gamma(\lambda+1)}{\pi^\lambda} \frac{J_{\frac{n}{2}+\lambda}(2\pi|x|)}{|x|^{\frac{n}{2}+\lambda}},$$

using again the identity in Appendix B.3.

B.6 Bessel Functions of Small Arguments

We seek the behavior of $J_k(r)$ as $r \to 0+$. We fix $\nu \in \mathbf{C}$ with $\operatorname{Re}\nu > -\frac{1}{2}$. Then we have the identity

$$J_\nu(r) = \frac{r^\nu}{2^\nu \Gamma(\nu+1)} + S_\nu(r),$$

where

$$S_\nu(r) = \frac{(r/2)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{+1} (e^{irt}-1)(1-t^2)^{\nu-\frac{1}{2}}\, dt$$

and S_ν satisfies

$$|S_\nu(r)| \le \frac{2^{-\operatorname{Re}\nu} r^{\operatorname{Re}\nu+1}}{(\operatorname{Re}\nu+\frac{1}{2})\,|\Gamma(\nu+\frac{1}{2})|\,\Gamma(\frac{1}{2})}.$$

To prove this estimate we note that

$$\begin{aligned} J_\nu(r) &= \frac{(r/2)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{+1} (1-t^2)^{\nu-\frac{1}{2}}\, dt + S_\nu(r) \\ &= \frac{(r/2)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^\pi (\sin^2\phi)^{\nu-\frac{1}{2}} (\sin\phi)\, d\phi + S_\nu(r) \\ &= \frac{(r/2)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \frac{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\nu+1)} + S_\nu(r), \end{aligned}$$

where we evaluated the last integral using the result in Appendix A.4. Using that $|e^{irt}-1| \le r|t|$, we deduce the assertion regarding the size of $|S_\nu(r)|$.