connection of this sort. The following is a general theorem saying that any vector-valued inequality is equivalent to some weighted inequality. The proof of the theorem is based on a minimax lemma whose precise formulation and proof can be found in Appendix H.

Theorem 7.5.8. *(a) Let $0 < p < q, r < \infty$. Let $\{T_j\}_j$ be a sequence of ~~sublinear~~ operators that map $L^q(\mu)$ to $L^r(\nu)$, where μ and ν are arbitrary measures. Then the vector-valued inequality*

$$\Big\|\Big(\sum_j |T_j(f_j)|^p\Big)^{\frac{1}{p}}\Big\|_{L^r} \le C\Big\|\Big(\sum_j |f_j|^p\Big)^{\frac{1}{p}}\Big\|_{L^q} \tag{7.5.23}$$

holds for all $f_j \in L^q(\mu)$ if and only if for every $u \ge 0$ in $L^{\frac{r}{r-p}}(\nu)$ there exists $U \ge 0$ in $L^{\frac{q}{q-p}}(\mu)$ with

$$\begin{aligned} \|U\|_{L^{\frac{q}{q-p}}} &\le \|u\|_{L^{\frac{r}{r-p}}}, \\ \sup_j \int |T_j(f)|^p\, u\, d\nu &\le C^p \int |f|^p\, U\, d\mu \quad \text{for all } f \in L^q(\mu). \end{aligned} \tag{7.5.24}$$

(b) Let $0 < q, r < p < \infty$, $r > \frac{p}{2}$, and let $\{T_j\}_j$ be as before. Then the vector-valued inequality (7.5.23) holds for all $f_j \in L^q(\mu)$ if and only if for every $u \ge 0$ in $L^{\frac{q}{p-q}}(\mu)$ there exists $U \ge 0$ in $L^{\frac{r}{p-r}}(\nu)$ with

$$\begin{aligned} \|U\|_{L^{\frac{r}{p-r}}} &\le \|u\|_{L^{\frac{q}{p-q}}}, \\ \sup_j \int |T_j(f)|^p\, U^{-1}\, d\nu &\le C^p \int |f|^p\, u^{-1}\, d\mu \quad \text{for all } f \in L^q(\mu). \end{aligned} \tag{7.5.25}$$

Proof. We begin with part (a). Given $f_j \in L^q(\mathbf{R}^n, \mu)$, we use (7.5.24) to obtain

$$\begin{aligned} \Big\|\Big(\sum_j |T_j(f_j)|^p\Big)^{\frac{1}{p}}\Big\|_{L^r(\nu)} &= \Big\|\sum_j |T_j(f_j)|^p\Big\|_{L^{\frac{r}{p}}(\nu)}^{\frac{1}{p}} \\ &= \sup_{\|u\|_{L^{\frac{r}{r-p}}} \le 1} \Big(\int_{\mathbf{R}^n} \sum_j |T_j(f_j)|^p\, u\, d\nu\Big)^{\frac{1}{p}} \\ &\le \sup_{\|u\|_{L^{\frac{r}{r-p}}} \le 1} C\Big(\int_{\mathbf{R}^n} \sum_j |f_j|^p\, U\, d\mu\Big)^{\frac{1}{p}} \\ &\le \sup_{\|u\|_{L^{\frac{r}{r-p}}} \le 1} C\Big\|\sum_j |f_j|^p\Big\|_{L^{\frac{q}{p}}(\mu)}^{\frac{1}{p}} \|U\|_{L^{\frac{q}{q-p}}}^{\frac{1}{p}} \\ &\le C\Big\|\Big(\sum_j |f_j|^p\Big)^{\frac{1}{p}}\Big\|_{L^q(\mu)}, \end{aligned}$$

which proves (7.5.23) with the same constant C as in (7.5.24). To prove the converse, we fix $u \in L^{\frac{r}{r-p}}(\nu)$, $u \geq 0$, with $\|u\|_{L^{\frac{r}{r-p}}} = 1$ and fix $f \in L^q(\mu)$. We define sets

$$A = \left\{a \in L^{\frac{q}{p}}(\mu) :\ a \geq |f|^p\right\}, \quad B = \left\{b \in L^{\frac{q}{q-p}}(\mu) :\ b \geq 0,\ \|b\|_{L^{\frac{q}{q-p}}} \leq 1\right\}.$$

Clearly A is a convex subset of $L^{\frac{q}{p}}$ and B is the unit ball of $L^{\frac{q}{q-p}} = (L^{\frac{q}{p}})^*$ which is weak* compact and convex. For a fixed j we define the bilinear functional Φ on $A \times B$ by

$$\Phi(a,b) = \int |T_j(f)|^p \, u \, d\nu - C^p \int a \, b \, d\mu \, .$$

Then $\Phi(\,\cdot\,,b)$ is linear (hence concave) on A for any $b \in B$. Moreover, for any $a \in A$, $\Phi(a,\,\cdot\,)$ is linear (hence convex) and continuous (hence lower semicontinuous) on B in the weak* topology, since if $b_k \to b$ in B, then

$$\int a \, b_k \, d\mu \to \int a \, b \, d\mu \quad \text{as } k \to \infty$$

for any $a \in L^{\frac{q}{p}}$. Thus the *minimax lemma* in Appendix H is applicable. This gives

$$\min_{b\in B} \sup_{a\in A} \Phi(a,b) = \sup_{a\in A} \min_{b\in B} \Phi(a,b) \, . \tag{7.5.26}$$

Hölder's inequality and the fact for all $a \in A$ we have $-a \leq -|f|^p$ yields

$$\begin{aligned}
\sup_{a\in A} \min_{b\in B} \Phi(a,b) &\leq \big\| |T_j(f)|^p \big\|_{L^{\frac{r}{p}}(\nu)} \|u\|_{L^{\frac{r}{r-p}}} - C^p \max_{b\in B} \int |f|^p \, b \, d\mu \\
&= \big\| T_j(f) \big\|_{L^r(\nu)}^p - C^p \big\| |f|^p \big\|_{L^{\frac{q}{p}}(\mu)} \leq 0,
\end{aligned}$$

by (7.5.23) applied to the sequence $(\cdots, 0, f, 0, \dots)$ with f in the j th spot. It follows from (7.5.26) that $\min_{b\in B} \sup_{a\in A} \Phi(a,b) \leq 0$. Thus there exists $U \in B$ (with $\|U\|_{L^{\frac{q}{p-q}}} \leq 1 = \|u\|_{L^{\frac{r}{p-r}}}$) such that $\Phi(a,U) \leq 0$ for every $a \in A$. Since $|f| \in A$ we have $\sup_j \Phi(|f|,U) \leq 0$ which yields (7.5.24) when $\|u\|_{L^{\frac{r}{r-p}}} = 1$. The general case follows by scaling u. This completes the proof of part (a).

The proof of part (b) is similar. Using the result of Exercise 7.5.1 and (7.5.25), given $f_j \in L^q(\mathbf{R}^n,\mu)$ we have

$$\begin{aligned}
\Big\| \big(\sum_j |f_j|^p \big)^{\frac{1}{p}} \Big\|_{L^q(\mu)} &= \Big\| \sum_j |f_j|^p \Big\|_{L^{\frac{q}{p}}(\mu)}^{\frac{1}{p}} \\
&= \inf_{\|u\|_{L^{\frac{q}{p-q}}} \leq 1} \left(\int_{\mathbf{R}^n} \sum_j |f_j|^p \, u^{-1} \, d\mu \right)^{\frac{1}{p}} \\
&\geq \frac{1}{C} \inf_{\|U\|_{L^{\frac{r}{p-r}}} \leq 1} \left(\int_{\mathbf{R}^n} \sum_j |T_j(f_j)|^p \, U^{-1} \, d\nu \right)^{\frac{1}{p}} \\
&= \frac{1}{C} \Big\| \sum_j |T_j(f_j)|^p \Big\|_{L^{\frac{r}{p}}(\nu)}^{\frac{1}{p}}
\end{aligned}$$

$$= \frac{1}{C}\left\|\Big(\sum_j |T_j(f_j)|^p\Big)^{\frac{1}{p}}\right\|_{L^r(\nu)}.$$

To prove the converse direction in part (b), given a fixed $u \geq 0$ in $L^{\frac{q}{p-q}}(\mu)$ with $\|u\|_{L^{\frac{q}{p-q}}} = 1$, we define sets and

$$A = \left\{a \in L^{\frac{q}{p}}(\mu) : \ a \geq |f|^p\right\}, \quad B = \left\{b \in L^{\frac{r}{p-r}}(\nu) : \ b \geq 0, \quad \|b\|_{L^{\frac{r}{p-r}}} \leq 1\right\}.$$

As $r > \frac{p}{2}$, $\frac{r}{p-r} > 1$ and so B is is weak* compact as the unit ball of $L^{\frac{r}{p-r}} = (L^{\frac{r}{2r-p}})^*$. We also define a functional Ψ on $A \times B$ by setting

$$\Psi(a,b) \ = \ \int |T_j(f)|^p b^{-1}\, d\nu - C^p \int a u^{-1}\, d\mu.$$

Then $\Psi(\,\cdot\,,b)$ is linear (hence concave) on A for any $b \in B$. Moreover, for any a in A, $\Psi(a,\cdot)$ is convex on B. To prove that Ψ is lower semi-continuous on B with respect to the weak* topology, we must show that the set $S = \{b \in B : \ \Psi(a,b) \leq K\}$ is closed in the weak* topology for any $K \in \mathbf{R}$. But S is convex, so this assertion is equivalent to that S is closed in the norm topology of $L^{\frac{r}{p-r}}(\nu)$. If $b_k \in B$ and $\|b_k - b\|_{L^{\frac{r}{p-r}}} \to 0$, then $b_{k_l} \to b$ a.e. for some subsequence. Fatou's lemma now implies $\Psi(a,b) \leq \liminf_{l\to\infty} b_{k_l}\Psi(a,b_{k_l}) \leq K$, thus $b \in S$ and so S is closed in the norm topology. The *minimax lemma* in Appendix H is applicable and yields (7.5.26) with Ψ in place of Φ. Hölder's inequality and Exercise 7.5.1 with $s = \frac{r}{p}$ yield

$$\begin{aligned}\sup_{a\in A}\min_{b\in B}\Psi(a,b) \ &\leq \big\||T_j(f)|^p\big\|_{L^{\frac{r}{p}}(\nu)} - C^p\int |f|^p u^{-1}\, d\mu \\ &\leq \|T_j(f)\|^p_{L^r(\nu)} - C^p\big\||f|^p\big\|_{L^{\frac{q}{p}}(\mu)} \leq 0,\end{aligned}$$

where the middle inequality is based on Exercise 7.5.1 with $s = \frac{q}{p}$ and the last one is a consequence of (7.5.23). Using (7.5.26), we obtain $\min_{b\in B}\sup_{a\in A}\Psi(a,b) \leq 0$, and this implies the existence of U in B (with $\|U\|_{L^{\frac{r}{p-r}}} \leq 1 = \|u\|_{L^{\frac{q}{p-q}}}$) such that $\Psi(a,U) \leq 0$ for all $a \in A$. Since $|f|$ lies in A we have $\sup_j \Psi(a,U) \leq 0$ which yields (7.5.25) when $\|u\|_{L^{\frac{q}{q-p}}} = 1$. The general case follows by scaling u. This completes the proof of part (b). □

Example 7.5.9. We use the previous theorem to obtain another proof of the vector-valued Hardy–Littlewood maximal inequality in Corollary 5.6.5. We take $T_j = M$ for all j. For given $1 < p < q < \infty$ and u in $L^{\frac{q}{q-p}}$ we set $s = \frac{q}{q-p}$ and $U = \|M\|^{-1}_{L^s\to L^s} M(u)$. In view of Exercise 7.1.7 we have

$$\|U\|_{L^s} \leq \|u\|_{L^s} \qquad \text{and} \qquad \int_{\mathbf{R}^n} M(f)^p u\, dx \leq C^p \int_{\mathbf{R}^n} |f|^p U\, dx.$$

Using Theorem 7.5.8, we obtain

$$\left\|\Big(\sum_j |M(f_j)|^p\Big)^{\frac{1}{p}}\right\|_{L^q} \leq C_{n,p,q}\left\|\Big(\sum_j |f_j|^p\Big)^{\frac{1}{p}}\right\|_{L^q} \tag{7.5.27}$$

whenever $1 < p < q < \infty$, an inequality obtained earlier in (5.6.25).

It turns out that no specific properties of the Hardy–Littlewood maximal function were used in the preceding inequality, and one could obtain a general result along these lines.