Since

$$\big\|T(f)\big\|_{L^{p,\infty}(w)} = \sup_{\lambda>0}\big\|T_\lambda(f)\big\|_{L^p(w)},$$

it follows that T maps $L^p(w)$ to $L^{p,\infty}(w)$ with the asserted norm. □

Assuming that the operator T in the preceding theorem is sublinear (or quasi-sublinear), we obtain the following result that contains a stronger conclusion.

Corollary 7.5.6. *Suppose that T is a sublinear operator on $\bigcup_{1\le q<\infty}\bigcup_{w\in A_q} L^q(w)$ that takes values in the space of measurable complex-valued functions. Fix $1\le p_0<\infty$ and suppose that there is an increasing function N on $[1,\infty)$ such that for all weights v in A_{p_0} we have*

$$\big\|T\big\|_{L^{p_0}(v)\to L^{p_0,\infty}(v)} \le N([v]_{A_{p_0}}). \tag{7.5.22}$$

Then for any $1<p<\infty$ and any weight w in A_p there is a constant $K'(n,p,p_0,[w]_{A_p})$ such that

$$\big\|T(f)\big\|_{L^p(w)} \le K'(n,p,p_0,[w]_{A_p})\big\|f\big\|_{L^p(w)}.$$

Proof. The proof follows from Theorem 7.5.5 and the Marcinkiewicz interpolation theorem. □

We end this subsection by observing that the conclusion of the extrapolation Theorem 7.5.3 can be strengthened to yield vector-valued estimates. This strengthening may be achieved by a simple adaptation of the proof discussed.

Corollary 7.5.7. *Suppose that T is defined on $\bigcup_{1\le q<\infty}\bigcup_{w\in A_q} L^q(w)$ and takes values in the space of all measurable complex-valued functions. Fix $1\le p_0<\infty$ and suppose that there is an increasing function N on $[1,\infty)$ such that for all weights v in A_{p_0} we have*

$$\big\|T\big\|_{L^{p_0}(v)\to L^{p_0}(v)} \le N\big([v]_{A_{p_0}}\big).$$

Then for every $1<p<\infty$ and every weight $w\in A_p$ we have

$$\Big\|\Big(\sum_j |T(f_j)|^{p_0}\Big)^{\frac{1}{p_0}}\Big\|_{L^p(w)} \le K(n,p,p_0,[w]_{A_p})\Big\|\Big(\sum_j |f_j|^{p_0}\Big)^{\frac{1}{p_0}}\Big\|_{L^p(w)}$$

for all sequences of functions f_j in $L^p(w)$, where $K\big(n,p,p_0,[w]_{A_p}\big)$ is as in Theorem 7.5.3.

Proof. To derive the claimed vector-valued inequality follow the proof of Theorem 7.5.3 replacing the function f by $(\sum_j |f_j|^{p_0})^{\frac{1}{p_0}}$ and $T(f)$ by $(\sum_j |T(f_j)|^{p_0})^{\frac{1}{p_0}}$. □

7.5.3 Weighted Inequalities Versus Vector-Valued Inequalities

We now discuss connections between weighted inequalities and vector-valued inequalities. The next result provides strong evidence that there is a nontrivial