We now proceed with the proof of the theorem. It is natural to split the proof into the cases $p < p_0$ and $p > p_0$.

Case (1): $p < p_0$. Assume momentarily that $v = R(f)^{-\frac{p_0}{(p_0/p)'}} w$ is an A_{p_0} weight. Then using (7.5.4) for this weight v we write

$$\begin{aligned}
&\big\|T(f)\big\|_{L^p(w)}^p \\
&= \int_{\mathbf{R}^n} |T(f)|^p R(f)^{-\frac{p}{(p_0/p)'}} R(f)^{\frac{p}{(p_0/p)'}} w\,dx \\
&\le \bigg(\int_{\mathbf{R}^n} |T(f)|^{p_0} R(f)^{-\frac{p_0}{(p_0/p)'}} w\,dx\bigg)^{\frac{p}{p_0}} \bigg(\int_{\mathbf{R}^n} R(f)^p\, w\,dx\bigg)^{\frac{1}{(p_0/p)'}} \\
&\le N\Big(\big[R(f)^{-\frac{p_0}{(p_0/p)'}} w\big]_{A_{p_0}}\Big)^p \bigg(\int_{\mathbf{R}^n} |f|^{p_0} R(f)^{-\frac{p_0}{(p_0/p)'}} w\,dx\bigg)^{\frac{p}{p_0}} \bigg(\int_{\mathbf{R}^n} R(f)^p\, w\,dx\bigg)^{\frac{1}{(p_0/p)'}} \\
&\le N\Big(\big[R(f)^{-\frac{p_0}{(p_0/p)'}} w\big]_{A_{p_0}}\Big)^p \bigg(\int_{\mathbf{R}^n} R(f)^{p_0} R(f)^{-\frac{p_0}{(p_0/p)'}} w\,dx\bigg)^{\frac{p}{p_0}} \bigg(\int_{\mathbf{R}^n} R(f)^p\, w\,dx\bigg)^{\frac{1}{(p_0/p)'}} \\
&= N\Big(\big[R(f)^{-\frac{p_0}{(p_0/p)'}} w\big]_{A_{p_0}}\Big)^p \bigg(\int_{\mathbf{R}^n} R(f)^p w\,dx\bigg)^{\frac{p}{p_0}} \bigg(\int_{\mathbf{R}^n} R(f)^p\, w\,dx\bigg)^{\frac{1}{(p_0/p)'}} \\
&\le N\Big(\big[R(f)^{-\frac{p_0}{(p_0/p)'}} w\big]_{A_{p_0}}\Big)^p \big(2\big\|f\big\|_{L^p(w)}\big)^p,
\end{aligned}$$

where we used Hölder's inequality with exponents p_0/p and $(p_0/p)'$, the hypothesis of the theorem, (7.5.7), and (7.5.8). Thus, we have the estimate

$$\big\|T(f)\big\|_{L^p(w)} \le 2N\Big(\big[R(f)^{-\frac{p_0}{(p_0/p)'}} w\big]_{A_{p_0}}\Big)\big\|f\big\|_{L^p(w)} \tag{7.5.13}$$

and it remains to obtain a bound for the A_{p_0} characteristic constant of $R(f)^{-\frac{p_0}{(p_0/p)'}}$. In view of (7.5.9), the function $R(f)$ is an A_1 weight with characteristic constant at most a constant multiple of $[w]_{A_p}^{\frac{1}{p-1}}$. Consequently, there is a constant C_1' such that

$$R(f)^{-1} \le C_1'\,[w]_{A_p}^{\frac{1}{p-1}} \bigg(\frac{1}{|Q|}\int_Q R(f)\,dx\bigg)^{-1}$$

for any cube Q in $\mathbf{R}^n$. Thus we have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q R(f)^{-\frac{p_0}{(p_0/p)'}} w\,dx \\
&\qquad \le \big(C_1'\,[w]_{A_p}^{\frac{1}{p-1}}\big)^{\frac{p_0}{(p_0/p)'}} \bigg(\frac{1}{|Q|}\int_Q R(f)\,dx\bigg)^{-\frac{p_0}{(p_0/p)'}} \bigg(\frac{1}{|Q|}\int_Q w\,dx\bigg).
\end{aligned} \tag{7.5.14}$$

Next we have

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \left(R(f)^{-\frac{p_0}{(p_0/p)'}}w\right)^{1-p_0'}dx\right)^{p_0-1}\\
&\qquad\qquad = \left(\frac{1}{|Q|}\int_Q R(f)^{\frac{p_0(p_0'-1)}{(p_0/p)'}}w^{1-p_0'}\,dx\right)^{p_0-1} \qquad (7.5.15)\\
&\qquad\qquad \le \left(\frac{1}{|Q|}\int_Q R(f)\,dx\right)^{\frac{p_0}{(p_0/p)'}}\left(\frac{1}{|Q|}\int_Q w^{1-p'}\right)^{p-1},
\end{aligned}$$

where we applied Hölder's inequality with exponents

$$\left(\frac{p'-1}{p_0'-1}\right)' \qquad \text{and} \qquad \frac{p'-1}{p_0'-1},$$

and we used that

$$\frac{p_0(p_0'-1)}{(p_0/p)'}\left(\frac{p'-1}{p_0'-1}\right)' = 1 \qquad \text{and} \qquad \frac{p_0-1}{\left(\frac{p'-1}{p_0'-1}\right)'} = \frac{p_0}{(p_0/p)'}.$$

Multiplying (7.5.14) by (7.5.15) and taking the supremum over all cubes Q in $\mathbf{R}^n$ we deduce that

$$\left[R(f)^{-\frac{p_0}{(p_0/p)'}}w\right]_{A_{p_0}} \le \left(C_1'\,[w]_{A_p}^{\frac{1}{p-1}}\right)^{\frac{p_0}{(p_0/p)'}}[w]_{A_p} = \kappa_1(n,p,p_0)\,[w]_{A_p}^{\frac{p_0-1}{p-1}}.$$

Combining this estimate with (7.5.13) and using the fact that N is an increasing function, we obtain the validity of (7.5.5) in the case $p < p_0$.

Case (2): $p > p_0$. In this case we set $r = p/p_0 > 1$. Then we have

$$\big\|T(f)\big\|_{L^p(w)}^p = \big\|\,|T(f)|^{p_0}\big\|_{L^r(w)}^r = \left(\int_{\mathbf{R}^n}|T(f)|^{p_0}h\,w\,dx\right)^r \qquad (7.5.16)$$

for some nonnegative function h with $L^{r'}(w)$ norm equal to 1. We define a function

$$H = \left[R'\left(h^{\frac{r'}{p'}}\right)\right]^{\frac{p'}{r'}}.$$

Obviously, we have $0 \le h \le H$ and thus

$$\begin{aligned}
\int_{\mathbf{R}^n}|T(f)|^{p_0}h\,w\,dx &\le \int_{\mathbf{R}^n}|T(f)|^{p_0}H\,w\,dx\\
&\le N\big([H\,w]_{A_{p_0}}\big)^{p_0}\big\|f\big\|_{L^{p_0}(H\,w)}^{p_0}\\
&\le N\big([H\,w]_{A_{p_0}}\big)^{p_0}\big\|\,|f|^{p_0}\big\|_{L^r(w)}\big\|H\big\|_{L^{r'}(w)} \qquad (7.5.17)\\
&\le 2^{\frac{p'}{r'}}N\big([H\,w]_{A_{p_0}}\big)^{p_0}\big\|f\big\|_{L^p(w)}^{p_0},
\end{aligned}$$

noting that

$$\|H\|_{L^{r'}(w)}^{r'} = \int_{\mathbf{R}^n} R'(h^{r'/p'})^{p'}\, w\, dx \le 2^{p'} \int_{\mathbf{R}^n} h^{r'}\, w\, dx = 2^{p'},$$

which is valid in view of (7.5.11). Moreover, this argument is based on the hypothesis of the theorem and requires that $H\,w$ be an A_{p_0} weight. To see this, we observe that condition (7.5.12) implies that $H^{r'/p'}w$ is an A_1 weight with characteristic constant at most a multiple of $[w]_{A_1}$. Thus, there is a constant C_2' that depends only on n and p such that

$$\frac{1}{|Q|}\int_Q H^{\frac{r'}{p'}}\, w\, dx \le C_2'\, [w]_{A_p} H^{\frac{r'}{p'}}\, w$$

for all cubes Q in $\mathbf{R}^n$. From this it follows that

$$(H\,w)^{-1} \le \kappa_2(n,p,p_0)\, [w]_{A_p}^{\frac{p'}{r'}} \left(\frac{1}{|Q|}\int_Q H^{\frac{r'}{p'}}\, w\, dx\right)^{-\frac{p'}{r'}} w^{\frac{p'}{r'}-1},$$

where we set $\kappa_2(n,p,p_0) = (C_2')^{p'/r'}$. We raise the preceding displayed expression to the power $p_0'-1$, we average over the cube Q, and then we raise to the power p_0-1. We deduce the estimate

$$\begin{aligned} &\left(\frac{1}{|Q|}\int_Q (H\,w)^{1-p_0'}\, dx\right)^{p_0-1} \\ &\qquad \le \kappa_2(n,p,p_0)\, [w]_{A_p}^{\frac{p'}{r'}} \left(\frac{1}{|Q|}\int_Q H^{\frac{r'}{p'}}\, w\, dx\right)^{-\frac{p'}{r'}} \left(\frac{1}{|Q|}\int_Q w^{1-p'}\, dx\right)^{p_0-1}, \end{aligned} \tag{7.5.18}$$

where we use the fact that

$$\left(\frac{p'}{r'}-1\right)(p_0'-1) = 1-p'.$$

Note that $r'/p' \ge 1$, since $p_0 \ge 1$. Using Hölder's inequality with exponents r'/p' and $(r'/p')'$ we obtain that

$$\frac{1}{|Q|}\int_Q H\,w\, dx \le \left(\frac{1}{|Q|}\int_Q H^{\frac{r'}{p'}}\, w\, dx\right)^{\frac{p'}{r'}} \left(\frac{1}{|Q|}\int_Q w\, dx\right)^{\frac{p_0-1}{p-1}}, \tag{7.5.19}$$

where we used that

$$\frac{1}{(\frac{r'}{p'})'} = \frac{p_0-1}{p-1}.$$

Multiplying (7.5.18) by (7.5.19), we deduce the estimate

$$\big[H\,w\big]_{A_{p_0}} \le \kappa_2(n,p,p_0)\, [w]_{A_p}^{\frac{p'}{r'}}\, [w]_{A_p}^{\frac{p_0-1}{p-1}} = \kappa_2(n,p,p_0)\, [w]_{A_p}.$$