Proof. We consider only the case $p < \infty$. First we note that convergence in $L^{p,q}$ implies convergence in measure. When $q = \infty$, this is proved in Proposition 1.1.9. When $q < \infty$, in view of Proposition 1.4.5 (16) and (1.4.7), it follows that

$$\sup_{t>0} t^{1/p} f^*(t) = \sup_{\alpha>0} \alpha d_f(\alpha)^{1/p} \le \left(\frac{q}{p}\right)^{1/q} \|f\|_{L^{p,q}}$$

for all $f \in L^{p,q}$, and from this it follows that convergence in $L^{p,q}$ implies convergence in measure.

Now let $\{f_n\}$ be a Cauchy sequence in $L^{p,q}$. Then $\{f_n\}$ is Cauchy in measure, and hence it has a subsequence $\{f_{n_k}\}$ that converges almost everywhere to some f by Theorem 1.1.13. Fix k_0 and apply property (9) in Proposition 1.4.5. Since $|f - f_{n_{k_0}}| = \lim_{k\to\infty} |f_{n_k} - f_{n_{k_0}}|$, it follows that

$$(f - f_{n_{k_0}})^*(t) \le \liminf_{k\to\infty} (f_{n_k} - f_{n_{k_0}})^*(t). \tag{1.4.10}$$

Raise (1.4.10) to the power q, multiply by $t^{q/p}$, integrate with respect to dt/t over $(0,\infty)$, and apply Fatou's lemma to obtain

$$\|f - f_{n_{k_0}}\|^q_{L^{p,q}} \le \liminf_{k\to\infty} \|f_{n_k} - f_{n_{k_0}}\|^q_{L^{p,q}}. \tag{1.4.11}$$

Now let $k_0 \to \infty$ in (1.4.11) and use the fact that $\{f_n\}$ is Cauchy to conclude that f_{n_k} converges to f in $L^{p,q}$. It is a general fact that if a Cauchy sequence has a convergent subsequence in a quasi-normed space, then the sequence is convergent to the same limit. It follows that f_n converges to f in $L^{p,q}$. □

Remark 1.4.12. It can be shown that the spaces $L^{p,q}$ are normable when p, q are bigger than 1; see Exercise 1.4.3. Therefore, these spaces can be normed to become Banach spaces.

It is well known that finitely simple functions are dense in L^p of any measure space, when $0 < p < \infty$. It is natural to ask whether finitely simple functions are also dense in $L^{p,q}$. This is in fact the case when $q \ne \infty$.

Theorem 1.4.13. *Finitely simple functions are dense in $L^{p,q}(X,\mu)$ when $0 < q < \infty$.*

Proof. Let $f \in L^{p,q}(X,\mu)$. Assume without loss of generality that $f \ge 0$. Since f lies in $L^{p,q} \subsetneqq L^{p,\infty}$ we have $\mu(\{f > \varepsilon\})^{1/p}\varepsilon \le (\frac{q}{p})^{\frac{1}{q}} \|f\|_{L^{p,q}} < \infty$ for every $\varepsilon > 0$ and consequently for any $A > 0$, $\mu(\{f > A\})$ is finite and tends to zero as $A \to \infty$. Thus for every $n = 1,2,3,\dots$, there is an $A_n > 0$ such that $\mu(\{f > A_n\}) < 2^{-n}$.

For each $n = 1,2,3,\dots$ define the function

$$\varphi_n(x) = \sum_{k=0}^{1+2^n A_n} \frac{k}{2^n} \chi_{\{k2^{-n} < f \le (k+1)2^{-n}\}} \chi_{\{2^{-n} < f \le A_n\}}.$$