and the *maximal operator* associated with T as follows:

$$T^{(*)}(f)(x) = \sup_{\varepsilon>0} \left| T^{(\varepsilon)}(f)(x) \right|.$$

We note that if T is in $CZO(\delta,A,B)$, then $T^{(\varepsilon)}(f)$ and $T^{(*)}(f)$ are well defined for all f in $\bigcup_{1\le p<\infty} L^p(\mathbf{R}^n)$. It is also well defined whenever f is locally integrable and satisfies $\int_{|x-y|\ge\varepsilon} |f(y)|\,|x-y|^{-n}dy < \infty$ for all $x \in \mathbf{R}^n$ and $\varepsilon > 0$.

The class of kernels in $SK(\delta,A)$ extends the family of convolution kernels that satisfy conditions (5.3.5) and (5.3.11). Obviously, the associated operators in $CZO(\delta,A,B)$ generalize the associated convolution operators.

A fundamental property of operators in $CZO(\delta,A,B)$ is that they have bounded extensions on all the $L^p(\mathbf{R}^n)$ spaces and also from $L^1(\mathbf{R}^n)$ to weak $L^1(\mathbf{R}^n)$. This is proved via an adaptation of Theorem 5.3.3; see Theorem 4.2.2 in [131]. There are analogous results for the maximal counterparts $T^{(*)}$ of elements of $CZO(\delta,A,B)$. In fact, an analogue of Theorem 5.3.5 yields that $T^{(*)}$ is L^p bounded for $1 < p < \infty$ and weak type $(1,1)$; this result is contained in Theorem 4.2.4 in [131].

We discuss weighted inequalities for singular integrals for general operators in $CZO(\delta,A,B)$. In Subsections 7.4.2 and 7.4.3, the reader may wish to replace kernels in $SK(\delta,A)$ by the more familiar functions $K(x)$ defined on $\mathbf{R}^n \setminus \{0\}$ that satisfy (5.3.5) and (5.3.11).

7.4.2 A Good Lambda Estimate for Singular Integrals

The following theorem is the main result of this section.

Theorem 7.4.3. *Let $1 \le p \le \infty$, $w \in A_p$, and T in $CZO(\delta,A,B)$. Then there exist positive constants*[1] *$C_0 = C_0(n,p,[w]_{A_p})$, $\varepsilon_0 = \varepsilon_0(n,p,[w]_{A_p})$, and $c_0(n,\delta)$, such that if $\gamma_0 = c_0(n,\delta)/A$, then for all $0<\gamma<\gamma_0$ we have*

$$w\big(\{T^{(*)}(f) > 3\lambda\} \cap \{M(f) \le \gamma\lambda\}\big) \le C_0 \gamma^{\varepsilon_0} (A+B)^{\varepsilon_0} w\big(\{T^{(*)}(f) > \lambda\}\big), \quad (7.4.6)$$

for all locally integrable functions f for which

$$\int_{|x-y|\ge\varepsilon} |f(y)|\,|x-y|^{-n}dy < \infty$$

for all $x \in \mathbf{R}^n$ and $\varepsilon > 0$. Here M denotes the Hardy–Littlewood maximal operator.

Proof. We write the open set

$$\Omega = \{T^{(*)}(f) > \lambda\} = \bigcup_j Q_j,$$

[1] the dependence on p is relevant only when $p < \infty$

where Q_j are the Whitney cubes (see Appendix J). We set

$$Q_j^* = 10\sqrt{n}\,Q_j\,,$$
$$Q_j^{**} = 10\sqrt{n}\,Q_j^*\,,$$

where aQ denotes the cube with the same center as Q whose side length is $a\ell(Q)$, where $\ell(Q)$ is the side length of Q. We note that in view of the properties of the Whitney cubes, the distance from Q_j to Ω^c is at most $4\sqrt{n}\,\ell(Q_j)$. But the distance from Q_j to the boundary of Q_j^* is $(5\sqrt{n}-\frac{1}{2})\,\ell(Q_j)$, which is bigger than $4\sqrt{n}\,\ell(Q_j)$. Therefore, Q_j^* must meet Ω^c and for every cube Q_j we fix a point y_j in $\Omega^c \cap Q_j^*$. See Figure 7.1.

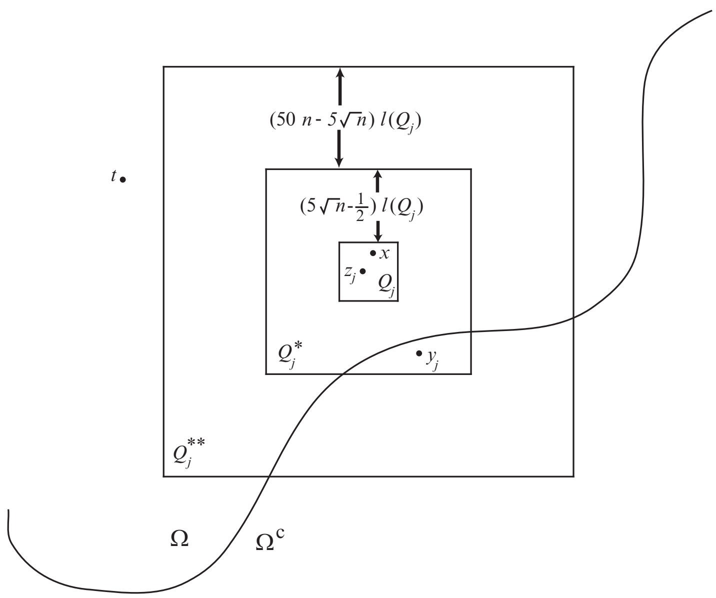

Fig. 7.1 A picture of the proof.

We also fix f satisfying the integrability condition of Theorem 7.4.3 and for each j we write $f = f_0^j + f_\infty^j$, where $f_0^j = f\chi_{Q_j^{**}}$ is the part of f near Q_j and $f_\infty^j = f\chi_{(Q_j^{**})^c}$ is the part of f away from Q_j. We now claim that the following estimate is true:

$$\big|Q_j \cap \{T^{(*)}(f) > 3\lambda\} \cap \{M(f) \le \gamma\lambda\}\big| \le C_n\,\gamma\,(A+B)\big|Q_j\big|\,. \tag{7.4.7}$$

Once the validity of (7.4.7) is established, we apply Theorem 7.3.3 (d) when $p=\infty$ or Proposition 7.2.8 when $p<\infty$ to obtain constants $\varepsilon_0, C_2 > 0$, which depend on $[w]_{A_p}$, p, n when $p<\infty$ and on $[w]_{A_\infty}$ and n when $p=\infty$, such that

$$w\big(Q_j \cap \{T^{(*)}(f) > 3\lambda\} \cap \{M(f) \le \gamma\lambda\}\big) \le C_2\,(C_n)^{\varepsilon_0}\,\gamma^{\varepsilon_0}\,(A+B)^{\varepsilon_0}\,w(Q_j)\,.$$