(f) There exist $p, C_3 < \infty$ such that $[w]_{A_p} \le C_3$. In other words, w lies in A_p for some $p \in [1,\infty)$.

All the constants $C_1, C_2, C_3, \alpha, \beta, \gamma, \delta, \alpha', \beta', \varepsilon, \varepsilon_0$, and p in (a)–(f) depend only on the dimension n and on $[w]_{A_\infty}$. Moreover, if any of the statements in (a)–(f) is valid, then so is any other statement in (a)–(f) with constants that depend only on the dimension n and the constants that appear in the assumed statement.

Proof. The proof follows from the sequence of implications

$$w \in A_\infty \implies (a) \implies (b) \implies (c) \implies (d) \implies (e) \implies (f) \implies w \in A_\infty .$$

At each step we keep track of the way the constants depend on the constants of the previous step. This is needed to validate the last assertion of the theorem.
$w \in A_\infty \implies (a)$

Fix a cube Q. Since multiplication of an A_∞ weight with a positive scalar does not alter its A_∞ characteristic, we may assume that $\int_Q \log w(t)\,dt = 0$. This implies that $\operatorname{Avg}_Q w \le [w]_{A_\infty}$. Then we have

$$\begin{aligned}
\big|\{x \in Q:\ w(x) \le \gamma \operatorname*{Avg}_Q w\}\big| &\le \big|\{x \in Q:\ w(x) \le \gamma [w]_{A_\infty}\}\big| \\
&= \big|\{x \in Q:\ \log(1+w(x)^{-1}) \ge \log(1+(\gamma[w]_{A_\infty})^{-1})\}\big| \\
&\le \frac{1}{\log(1+(\gamma[w]_{A_\infty})^{-1})} \int_Q \log \frac{1+w(t)}{w(t)}\,dt \\
&= \frac{1}{\log(1+(\gamma[w]_{A_\infty})^{-1})} \int_Q \log(1+w(t))\,dt \\
&\le \frac{1}{\log(1+(\gamma[w]_{A_\infty})^{-1})} \int_Q w(t)\,dt \\
&\le \frac{[w]_{A_\infty}|Q|}{\log(1+(\gamma[w]_{A_\infty})^{-1})} \\
&= \frac{1}{2}|Q|,
\end{aligned}$$

which proves (a) with $\gamma = [w]_{A_\infty}^{-1}(e^{2[w]_{A_\infty}} - 1)^{-1}$ and $\delta = \frac{1}{2}$.
$(a) \implies (b)$

Let Q be fixed and let A be a subset of Q with $w(A) > \beta w(Q)$ for some β to be chosen later. Setting $S = Q \setminus A$, we have $w(S) < (1-\beta)w(Q)$. We write $S = S_1 \cup S_2$, where

$$S_1 = \{x \in S:\ w(x) > \gamma \operatorname{Avg}_Q w\} \quad \text{and} \quad S_2 = \{x \in S:\ w(x) \le \gamma \operatorname{Avg}_Q w\}.$$

For S_2 we have $|S_2| \le \delta|Q|$ by assumption (a). For S_1 we use Chebyshev's inequality to obtain

$$|S_1| \le \frac{1}{\gamma \operatorname*{Avg}_Q w} \int_S w(t)\,dt = \frac{|Q|}{\gamma}\frac{w(S)}{w(Q)} < \frac{1-\beta}{\gamma}|Q|.$$

Adding the estimates for $|S_1|$ and $|S_2|$, we obtain

$$|S| \le |S_1| + |S_2| < \frac{1-\beta}{\gamma}|Q| + \delta|Q| = \Big(\delta + \frac{1-\beta}{\gamma}\Big)|Q|\,.$$

Choosing numbers α, β in $(0,1)$ such that $\delta + \frac{1-\beta}{\gamma} = 1-\alpha$, for example $\alpha = \frac{1-\delta}{2}$ and $\beta = 1 - \frac{(1-\delta)\gamma}{2}$, we obtain $|S| < (1-\alpha)|Q|$, that is, $|A| > \alpha|Q|$.

$(b) \implies (c)$

This was proved in Corollary 7.2.4. To keep track of the constants, we note that the choices

$$\varepsilon = \frac{-\frac{1}{2}\log\beta}{\log 2^n - \log\alpha} \qquad \text{and} \qquad C_1 = 1 + \frac{(2^n\alpha^{-1})^\varepsilon}{1-(2^n\alpha^{-1})^\varepsilon\beta}$$

as given in (7.2.6) and (7.2.7) serve our purposes.

$(c) \implies (d)$

We apply first Hölder's inequality with exponents $1+\varepsilon$ and $(1+\varepsilon)/\varepsilon$ and then the reverse Hölder estimate to obtain

$$\begin{aligned}
\int_A w(x)\,dx &\le \Big(\int_A w(x)^{1+\varepsilon}\,dx\Big)^{\frac{1}{1+\varepsilon}} |A|^{\frac{\varepsilon}{1+\varepsilon}} \\
&\le \Big(\frac{1}{|Q|}\int_Q w(x)^{1+\varepsilon}\,dx\Big)^{\frac{1}{1+\varepsilon}} |Q|^{\frac{1}{1+\varepsilon}}\,|A|^{\frac{\varepsilon}{1+\varepsilon}} \\
&\le \frac{C_1}{|Q|}\int_Q w(x)\,dx\,|Q|^{\frac{1}{1+\varepsilon}}\,|A|^{\frac{\varepsilon}{1+\varepsilon}}\,,
\end{aligned}$$

which gives

$$\frac{w(A)}{w(Q)} \le C_1\Big(\frac{|A|}{|Q|}\Big)^{\frac{\varepsilon}{1+\varepsilon}}\,.$$

This proves (d) with $\varepsilon_0 = \frac{\varepsilon}{1+\varepsilon}$ and $C_2 = C_1$.

$(d) \implies (e)$

Pick an $0 < \alpha'' < 1$ small enough that $\beta'' = C_2(\alpha'')^{\varepsilon_0} < 1$. It follows from (d) that

$$|A| \le \alpha''|Q| \implies w(A) \le \beta'' w(Q) \tag{7.3.5}$$

for all cubes Q and all A measurable subsets of Q. Replacing A by $Q\setminus A$, the implication in (7.3.5) can be equivalently written as

$$|A| \ge (1-\alpha'')|Q| \implies w(A) \ge (1-\beta'')w(Q)\,.$$

In other words, for measurable subsets A of Q we have

$$w(A) < (1-\beta'')w(Q) \implies |A| < (1-\alpha'')|Q|\,, \tag{7.3.6}$$